TEACHING ARITHMETIC

IN THE

PRIMARY GRADES

BY

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SILVER, BURDETT AND COMPANY

NEW YORK NEWARK BOSTON CHICAGO SAN FRANCISCO

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Printed in the United States of America

PREFACE

Within the last decade some two dozen books on the teaching of arithmetic have appeared and in addition a great many courses of study, teachers' manuals, bulletins, and articles in educational periodicals have been published, all in an effort to improve arithmetic teaching. There can be no doubt that each of these has had an influence, sometimes large and sometimes small. Although in many instances the thinking of teachers and supervisors has been modified, and their practices have undergone change, nevertheless many of the educational practices of some of us and some of the educational practices of all of us are essentially what they were a year ago, a decade ago, or a generation ago.

Today, however, educational practices, curriculum materials, and teaching methods are being subjected to minute scrutiny, and must be demonstrably good if they are to be permitted to remain. Individuals and groups of individuals are studying arithmetic. Extensive researches have been completed or are under way at the University of Chicago, the State University of Iowa, Ohio State University, in Michigan, in Wisconsin, and elsewhere. Enough has been done to suggest that the teaching procedures which have been accepted without question have not been yielding satisfactory results and must give way to other procedures which have been demonstrated to be superior or which seem to accord better with accepted principles of educational psychology.

So voluminous is the available material on arithmetic teaching and so many are the details that we can no longer expect to discuss "The Teaching of Arithmetic" in a single volume. Accordingly, I have dealt with the problems of arithmetic teaching in grades four, five, and six in *Teaching Arithmetic in the Intermediate Grades*. The present volume covers the work of the

primary grades, or grades one, two, and three. Later volumes will deal with the teaching of mathematics in junior and senior high schools.

The material in this book was originally prepared to serve as a text for my course, *Teaching Arithmetic in the Primary Grades*. It has been used in mimeographed form in my classes at Ohio University, in the Athens public schools as a handbook for teachers, in several of the county normal schools of Ohio, and in many other Ohio communities to which it was sent in response to requests from individual teachers. Thus, an opportunity was afforded to subject the material to the severe test of classroom usableness. In so far as practicable the suggestions made by these users were incorporated in the present volume.

In this book, I have attempted to bring together the results of scientific experimentation and to blend with these the principles of a sound educational psychology in such a form that there might be available for the primary teacher rather detailed and very specific suggestions for planning and teaching the early lessons in arithmetic. Seven years of teaching and supervisory experience in public schools, three years as Extension lecturer at Ohio University, and six years as head of the Department of Mathematics in the College of Education have convinced me that teachers not only need but also desire specific suggestions. These suggestions need not be of the rule-of-thumb variety but must be more than vague and generalized statements, dealing superficially with teaching problems.

In general, books dealing with the teaching of arithmetic have given scant attention to the teaching of this subject in grades one, two, and three. The suggestions devoted specifically to the problems involved are usually so brief and of such a general nature that they cannot be of very great assistance to the primary teacher.

One can readily see why books written a few years ago should have but little to offer on arithmetic in the primary grades. Comparatively little was known about this subject; researches

which have since been reported were just getting under way at that time. Today, many problems remain to be solved; we must still build many of our teaching practices upon theory rather than upon ascertained fact. But enough is known to justify a book which, in no small way, shall base its recommendations upon the outcome of research.

After an introductory chapter in which certain trends in present day arithmetic teaching are briefly stated, Chapter II considers the much disputed question as to whether a definite program of arithmetic instruction should be provided in the first grade. Chapter III is devoted to the teaching of counting and reading and writing numbers and the development of number concept. Chapters IV, V, and VI present in some detail elementary work in addition and subtraction. In a very similar manner, Chapters VII, VIII, and IX deal with multiplication and division so far as these subjects are offered in the curriculum of the primary grades. Chapter X, The Solution of Problems, is intended to make a much more prominent place for a phase of arithmetic work which is often neglected in the primary grades. Finally, Chapter XI gives consideration to certain major problems in the construction of a course of study and suggests, in some detail, the items which should make up the course for each of the three primary grades.

The author is indebted to a great many people for valuable suggestions and helpful criticisms. He is particularly obligated to the several thousand students who have taken his courses; among these have been many who could speak and write in a most stimulating fashion. Several of his associates at Ohio University have read the manuscript and made valuable suggestions; of these particular acknowledgment should be made to Miss Edith Beechel, Assistant Director of Teacher Training, and to Miss Jeannette Smith, First Grade Critic Teacher in the Training School. John R. Patterson, Superintendent of schools, Athens, O., has also read the manuscript and offered valuable suggestions.

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TEACHING ARITHMETIC
IN THE PRIMARY GRADES

Chapter I

INTRODUCTION

Arithmetic is a subject of long standing in the elementary school. In the days when courses were confined more or less closely to the three R's, which perhaps represents the most primitive degree of simplicity in American elementary curricula, arithmetic stood out as a prominent member of the triumvirate. Less than a century ago, in the middle western states, to have "ciphered through to the rule of three" was an academic distinction of no mean import, comparable, perhaps, to the ability to spell all of the words in McGuffey's Eclectic Speller.

At first a practical subject. History tells us that arithmetic was originally taught primarily because of its practical value. According to Smith,[1] it was found in the early Chinese curricula in order that the boys might have sufficient knowledge of the four fundamental processes for the common vocations of life. And so it has remained in the far East for two thousand years. The trade revival of the Middle Ages was accompanied by a corresponding revival in commercial arithmetic. From Italy, the gateway between Europe and the Orient, the practical demands for arithmetic spread north and west through Europe and finally to American shores. The Church schools were slow to meet this practical demand, concerning themselves more with mystical and formalistic doctrines and topics. Although using

[1] Smith, David Eugene. *The Teaching of Elementary Mathematics.* The Macmillan Company, 1916, p. 2.

the subject to some extent as a "whetstone of wit," they developed arithmetic but little beyond what was necessary for keeping the calendar straight. Accordingly, a little knowledge of, and skill in, the subject distinguished one; ability to manipulate simple fractions qualified one as a mathematician. And, as already indicated, in this country less than a century ago, to have attained to a mastery of simple proportion meant that in the ordinary community there were few, if any, to whom one must look as his superiors in this field.

Faculty psychology and mental discipline. The spread of the influence of faculty psychology and its offspring, general discipline, have, however, in comparatively recent times, greatly changed the content of the curriculum in arithmetic. This influence is readily seen in the primary grades but was much more potent in the intermediate and upper grades. The subject became of value for more than practical reasons. It became the function of the school to develop in children the faculties of memory, imagination, reasoning, etc., with the thought that, once developed, they could be applied successfully to any situation whatever in the handling of which these functions were involved. Arithmetic was considered particularly appropriate for this purpose because of (1) its precision and elegance and (2) its difficulty. Accordingly, the curriculum broke loose from its practical moorings and drifted far on the sea of unreal and impossible problem situations. Whereas, heretofore, the farmer had ascertained the number of sheep he possessed by counting, he now found it necessary, in order to arrive at an answer to a question pertaining to the size of his flock, to consider that if to the square root of the number in his flock, eight are added and the result multiplied by four, the product would be the same as the number in his flock. In other words, he must solve the equation $4(\sqrt{X} + 8) = X$, without the X or other literal symbol (for algebra was a tardy arrival in alleviating the suffering of shipwrecked and stranded students in arithmetic) or as a process in pure mental gymnastics.

Modern tendency toward the practical, or social. However, the modern tendency, as shown particularly in the last decade, is distinctly away from treating arithmetic as a disciplinary subject primarily, and back to its *practical*, or, as we now choose to say, *social* basis. Under the leadership of men like Thorndike, Wilson, and Stone we are critically examining the elements of the curriculum in arithmetic with this question in mind, What will this material contribute toward enabling the pupil to function effectively as a member of modern society? Elementary teachers, particularly those in the primary grades, are little interested in the "mental discipline" controversy but much concerned about discovering the most effective means of teaching children the essential elements of this subject.

There is much difference of opinion among educational authorities as to how prominent a place arithmetic should receive in the primary grades. Some would provide a heavy program of arithmetic instruction for the first grade, pushing rapidly ahead to difficult examples in multiplication and division in grades two and three. Others, believing that the first grade pupil's energies should be largely devoted to learning to read, would make no provision for arithmetic in the first grade course of study, nor would they accord it a prominent place until grade three or even grade four is reached.

Children's everyday activities involve number. Since arithmetic is studied primarily for its practical value, however, even young children should be acquainted with some of its elements. The everyday activities of seven- and eight-year-old children may quite well reveal the necessity of some skill in the fundamental operations with whole numbers. Children of these ages in their games and plays, and in their activities in which they earn, spend, and save money, will count, add and subtract, and, sometimes, multiply and divide. These activities are encouraged in many schools by thrift campaigns and school savings banks.

Inasmuch as the ordinary activities of children precipitate them into situations involving some use of number skills, the

teacher is at once concerned with the problem of forming proper fundamental habits. She knows full well that if a pupil is forced to rely upon himself in devising means for solving his problems involving numbers, or to appeal to his parents who are probably unacquainted with good methods of teaching, improper habits will be formed. These improper habits mean slow and inaccurate work; they, furthermore, hinder the pupil in learning proper methods of procedure later.

Students of primary education have quite generally concluded, then, that arithmetic should be given a very definite place in the primary grades. If the subject is entirely ignored by the school, other and much less efficient instructional facilities may be utilized by the inquisitive learner. If it is left to a merely "incidental" place and treatment, the amount of attention devoted to the subject will probably depend largely upon the teacher's interest in it, her aggressiveness, and the reserve of time and energy which she feels that she can devote to it. A recognition of the value of arithmetic skills, and of the importance of acquiring them properly, will persuade one that the subject must find a very definite place in the primary curriculum and that it cannot be left to merely incidental consideration.

To contend that a very definite place should be made for arithmetic in the curriculum of the primary grades, however, is not to say that the requirements should be heavy and elaborate and the instruction rigidly formal. *The amount of attention given to arithmetic in the primary grades should depend largely upon the pupil's needs for number skills as shown by an analysis of his activities in and out of school.*

Variations in practice. As to the extent of these needs, there is little scientific evidence available. The activities of children have not been sufficiently studied and recorded to enable us to know just how much arithmetical instruction should be provided for the average first grade, second grade, or third grade. If we examine courses of study we find interesting and extensive variation. For example, in the recommendations of state courses

of study as regards reading and writing numbers, we find Colorado setting 200 as the limit for the second grade, while Oregon recommends that second grade pupils be taken as far as 10,000. In the third grade, Vermont would go to 10,000, while Georgia would have third grade children read and write numbers as far as 1,000,000. Frequently, state courses of study mention reading and writing numbers but specify no limits for this work, so variations in practice are probably greater than has been suggested.

Marked variation in course of study requirements in notation and numeration is also found among cities. In the first grade, Moline, Illinois, would go to 20, while Berkeley, California, would take the pupils to 132 (the number of pages in a text in reading). In the second grade, Owensboro, Kentucky, goes as far as 100, while Houston, Texas, recommends that pupils in this grade be taught to read and write numbers to 9,999. In the third grade, Paterson, New Jersey, specifies 5,000 as the limit, but Bellingham, Washington, would have these pupils instructed as far as 1,000,000.

Roman notation. Similar variations are found in the amount of Roman notation taught in the primary grades. The Kansas course of study specifies that Roman notation shall be taught to XXI in the first grade and to L in the second grade, while the Colorado course does not mention the subject in grades one and two and recommends that it be taught as far as V in grade three. El Paso, Texas, teaches this subject to XX in grade three without giving it any attention in the first and second grades, but Norwalk, Connecticut, carries instruction in this subject to LX in the second grade and to M in the third grade.[2]

Large differences are found in course of study requirements in other topics commonly taught in the primary grades. Many schools teach none of the addition and subtraction combina-

[2] The statements of this and the preceding paragraphs are based upon an analysis of printed courses of study. In some cases, these may not have been the most recent courses issued but all were published within the last six or seven years.

tions in the first grade and a few postpone all instruction in these operations until grade three is reached. Other schools, however, teach all of the fundamental combinations of addition and subtraction in the first grade and a number of the multiplication and division combinations as well. In one school, the only arithmetic instruction which first grade children receive is that which occurs quite incidentally (or accidentally), independent of any definite plan which the teacher may have; in another school, arithmetic instruction in this grade is quite formal and the course of study requirements are heavy and rigid.

After reading the preceding discussion, the reader will probably wonder just what does constitute good practice as regards teaching procedure and course of study making in arithmetic in the primary grades; just what is the place of arithmetic in the primary curriculum? Unfortunately, no one seems to know fully and completely the answer to this question. Much research remains to be done before we can say what topics should be treated in the curriculum of the primary grades, in which grades each of these topics should be located, how extensive the treatment of a topic should be, and what constitutes the best teaching procedure. Here and there bits of research work have been done which throw some light on these questions. Where there is no objective evidence from research studies, we can but set up what seems to be good theory, founded upon the principles of a sound educational psychology.

In later chapters, such fact and such theory will be organized for the primary teacher's guidance and convenience. In the meantime, let us turn to a question which is the cause of much dispute among educators—How much arithmetic should be taught in the first grade?

EXERCISES

1. What were the "three R's"?

2. Why should arithmetic have been a prominent subject in early curricula?

3. In what way is the demand for arithmetic skills dependent upon the commercial development of a country?

4. What is faculty psychology? What is mental discipline? To what extent does training in problem solving enable one to reason more effectively in other situations?

5. Find problems in arithmetic texts which show the surviving influence of the doctrine of general discipline.

6. How would you decide the amount of arithmetic instruction to be provided in the first grade?

7. What is meant by the "incidental" teaching of arithmetic?

8. Examine courses of study for variations in requirements as to (*a*) the topics taught and (*b*) the extent to which instruction in a given topic is carried.

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Chapter II

ARITHMETIC IN THE FIRST GRADE

Ten or fifteen years ago, much was said and written concerning the inadvisability of teaching arithmetic in the first grade. Burnham[1] contended on hygienic grounds that formal instruction in arithmetic began too early. After considerable argument on the subject, illustrated by a number of concrete instances, he concludes his discussion as follows:

"Formal instruction in this subject probably should not be begun until the age of eight or ten. Whatever arithmetical work is done before this age should be spontaneous on the part of the pupil. By postponing the instruction until this age it would probably be possible in most cases to do away with the artificial devices and methods liable to cause arrests or interference of association. The work at first should be simple and concrete, and difficult processes should be postponed until a later period. Processes involving logic should not be given until the pupil has acquired a rich store of concrete experience. Special care should be taken in the case of nervous children that no tasks or methods employed should be the occasion of worry or of the development of neuroses like chorea. Special attention should be given in general to the secondary effects of instruction important from the point of view of mental hygiene."[2]

An experimental study. Taylor[3] reports an interesting experiment which he conducted in one of the New York City

[1] Burnham, Wm. M. "Arithmetic and School Hygiene." *Pedagogical Seminary*, XVIII: 54-73, March, 1911.

[2] *Ibid.*, p. 72.

[3] Taylor, Joseph S. "Omitting Arithmetic in the First Year." *Educational Administration and Supervision*, II: 87-93, February, 1916.

schools during the years 1913-1915. Two first grade classes were permitted to omit all arithmetic work except counting, and to devote the extra time to English. "The first two classes upon which the experiment was tried have just completed the 2B grade. They began arithmetic work upon entering the 2A grade, and in twenty weeks they completed 1A and 1B arithmetic work and all of the 2A arithmetic. In the 2B grade this term, they did the regular 2B work in arithmetic. . . . The experiment has been successful because, while the quality and quantity of the reading has improved, the number work has apparently suffered no harm." [4]

Summarizing the results of his experiment, Taylor says,—"It is thus shown that the classes which omitted number work during the first year of school not only held their own against those that had number, but actually outstripped their competitors. Hence it is hard to escape the conclusion that the time now given to arithmetic in the first year is worse than wasted; for when that time was devoted to English . . . the children were able to read about three times as much matter as classes of the same grade where arithmetic was studied. At the same time, there was no loss of arithmetical ability, since the children were able easily to make up during the second year the little arithmetic they had missed during the first year." [5] Thus, it seems that, because of their greater maturity and richer experiences, second grade pupils do easily the arithmetic work ordinarily required in the first grade and the work of the second grade as well. (The reader will note that in New York 1A refers to the lower half of the first grade work and 1B to the upper half, etc.)

Out-of-school uses of arithmetic. When one studies the out-of-school activities of first grade children, however, he can readily find a large number of instances in children's experiences in which the elements of arithmetic are conspicuous. They make their own purchases in stores; they play games involving count-

[4] Taylor, Joseph S., *loc. cit.*, p. 87.
[5] *Ibid.*, p. 93.

ing; they read Arabic numerals in finding pages of books and Roman numerals in telling time; they divide candy, fruit and other articles with brothers, sisters and playmates; they play store; they deposit money in toy banks at home and in school banks; and they engage in many other activities in which numbers and number operations are used. Miss Nila B. Smith,[6] in studying the uses of arithmetic in the lives of first grade children, found the activities which we have mentioned and many others of less frequent occurrence. Table I is taken from the account of her investigation and shows the different activities discovered and the frequency of each.

TABLE I. *Relative Frequency with Which Situations Involving Arithmetic Occurred* (After Smith)

Activity	*Per Cent*
Transactions carried on in stores	30.0
Games involving counting	18.0
Reading Roman numerals on the clock	14.0
Reading Arabic numerals in finding pages in books	13.0
Dividing food with playmates and pets (fractions)	6.0
Depositing money in and drawing money from toy banks	5.0
Playing store	3.0
Measuring distance	2.2
Using calendars	2.0
Running errands	1.2
Setting the table	1.2
Buying and selling tickets	1.1
Acting as newsboy	1.0
Measuring in sewing	1.0
Counting in rhymes and jingles	0.5
Reading house numbers	0.2
Investments (made for them)	0.1
Measuring in manual training	0.1
Measuring height	0.1
Measuring objects	0.1
Reading numbers on hooks in hall	0.1
Reading numbers on tickets	0.1
Total	100.0

[6] Smith, Nila B. "An Investigation of the Uses of Arithmetic in the out-of-school Life of First-Grade Children." *The Elementary School Journal*, XXIV: 621-626.

It will be seen that 30 per cent of the arithmetic activities of these children had to do with transactions carried on in stores, 18 per cent were connected with games involving counting, etc.

When the data of Table I were tabulated according to the type of operation involved, Table II was the result. This table lists the processes which occurred and gives the relative frequency of each. Thus, in 35 per cent of the cases, the process used was addition, etc.

TABLE II. *Relative Frequency of Arithmetical Operations* (After Smith)

Process	*Per Cent*
Addition	35.0
Counting	23.0
Subtraction	12.0
Fractions	8.0
Reading Arabic numerals	6.0
Measuring	5.5
Comparison	4.3
Reading Roman numerals	3.1
Multiplication	2.0
Division	1.1
Total	100.0

In summarizing her study, Miss Smith says: "This study furnishes data of considerable importance in solving several problems concerning arithmetic instruction: (1) Which arithmetical processes should receive emphasis in the first grade in order to enable the children to meet the arithmetical needs in their everyday lives? (2) What types of concrete situations should be provided for the children in the first grade that they may have practice in making applications similar to those required of them outside of school? Incidentally, the study throws light on the controversy as to whether counting or spatial measurement should be the starting-point in instructing primary pupils in arithmetic. From these data it would appear that counting plays a very large part and that spatial measurement plays a very small part in the ordinary uses of number made by first grade pupils."[7]

[7] Smith, Nila B., *loc. cit.*, p. 626.

Arithmetic in the kindergarten. Kindergartners and first-grade teachers have been disposed to recognize the fact that their charges have real and numerous out-of-school uses for number by providing a modicum of number experiences for kindergarten children and a very definite place for this subject in the curriculum of the first grade. In a bulletin prepared by a subcommittee of the Bureau of Education Committee of the International Kindergarten Union,[8] Miss Alice L. Harris, assistant superintendent of schools, Worcester, Massachusetts, discusses briefly the place of number in the kindergarten first grade curriculum and makes recommendations for a course of study.

"All kindergartners seem to agree on the following quantitative experiences of kindergarten children:"

1. Counting:
   To find the number of children in the circle, chairs in the circle or room, blocks used, objects constructed, objects seen on walks, pennies brought in for savings stamps, squares in the 16 square fold, etc. This counting may sometimes be carried on to 100.
2. Construction:
   Oral expression about the work in construction provides many opportunities for language training in relation to number. The activities as suggested in the Kindergarten Curriculum would afford elementary knowledge of the facts listed under the following headings:
3. Fractions:
   Wholes, halves, and possibly quarters.
4. Measurement:
   Length—long or short.
   Width—wide or narrow, thick or thin.
   Units—cubes, or bricks when used in building.
5. Games emphasizing number:
   Muffin Man (1 and 1 more, 2 and 2 more, 4 and 4 more, "So 8 of us know the Muffin Man").
   Chickadees (5—"1 flew away and then there were 4," etc.).

[8] *A Kindergarten-First-Grade Curriculum.* Department of the Interior, Bureau of Education, Bulletin, 1922, No. 15.

Little Indians—(10).
Family, members—parents, children, brothers, and sisters.
Sense games—feeling the number of objects.

6. Time.
From clock—short time, long time; how clock looks at 9 o'clock.
From calendar—days of week.
7. Proportion—in industrial arts.
Chimney too large for house.
Furniture too small for room.
8. Recognition of number.
Groups—4 children here, etc.[9]

With this background of quantitative experiences in the kindergarten, it would be strange, indeed, if no provision was made for further quantitative experiences and for number instruction in the first grade. To be sure, arithmetic instruction in the first grade should not be formal. All present-day students of primary education seem to agree upon this. The basis for such arithmetic instruction as is given "should be the child's experiences and the knowledge of number obtained through the activities of home, school, neighborhood, and situations arising through these. While for the child it should appear incidental to these situations, *it should be definite in the mind of the teacher*."[10]

The course in the first grade. The following is Miss Harris' recommendation for attainments by the end of the first school year.[11]

1. *Attitudes, interests, tastes.* At the close of the first year children should have an appreciation of the fact that arithmetic is not only skill in calculation, but that it satisfies the quantitative needs of life. This attainment is based on the supposition

[9] *A Kindergarten-First-Grade Curriculum, op. cit.*, pp. 52-53
[10] *Ibid.*, p. 53. (Italics supplied).
[11] We are quoting Miss Harris' recommendations but not fully endorsing them. In Chapter XI we have outlined what seems to be a desirable course of study.

that throughout the year they have had abundant opportunity to express through imitation their intrinsic interest in the institutional occupations of their elders.

2. *Habits, skills.* At the close of the first year children should have acquired—

A degree of self-dependence and initiative through having worked with things in expressing individual or group ideas.
Familiarity with the foot rule, and ability to draw and measure relatively accurately with it.
Ability to estimate length and width of object in a limited way.
Ability to follow simple printed directions: such as, make a 2-inch square; cut 4 inches of string; cut two 12-inch strips, etc.
Ability to follow simple printed directions, using such words as measure, fold, paste, count, build, etc.

3. *Knowledge, information.* At the close of the first year children should be able—

To count by 1's, 2's, 10's.
To count backward from 20.
To group objects—five 2's, six 10's, etc.
To recognize small groups without counting, e.g., 2 marbles, 3 sticks.
To recognize larger groups in a "domino" or uniform arrangement.
To recognize a cent, nickel, dime, quarter, half dollar, and a bill.
To know value of nickel, dime, quarter.
To recognize numbers 1 to 100.
To write them from dictation.
To recognize the Roman numerals I to XII.
To know through experience the meaning of ⅓, ½, ¼, $-$, $+$, $=$.[12]

McFarland[13] contends eloquently for a more prominent place for arithmetic in the first grade. Citing the report of the N. E. A.

[12] *Ibid.*, p. 56.
[13] McFarland, Blanche Bulifant. "A Plea for Arithmetic in the First Grade." *The Kindergarten and First Grade*, VII: 1-7, 49-55, 92-98, 137-142, 180-185, January, February, March, April, and May, 1922.

Committee of Fifteen (1895) to the effect that "with the right methods, and a wise use of time in preparing the arithmetic lesson in and out of school, five years are sufficient," she shows that the result was the elimination of arithmetic from grade one and, sometimes, from grade two, except incidental teaching. The "incidental teaching" recommended for those first and second grades in which required work in arithmetic was omitted she found to be a decidedly variable quantity, differing with each teacher, frequently becoming "accidental teaching," and with many teachers practically omitted altogether. Miss McFarland insists that "incidental teaching" means that the first grade teacher has no definite goal toward which she may work and that second and third grade teachers do not know what to expect of the pupils coming to them. The modern, social, purged course is not too difficult for first grade pupils. Furthermore, they have a background of experiences which justifies a definite program of number instruction.

The modern first grade teacher provides very definitely for instruction in arithmetic, but she makes that instruction a series of interesting and delightful experiences and does not permit it to interfere with the principal business of the first grade—teaching children to read.

EXERCISES

1. On what grounds did Burnham argue for the late introduction of arithmetic?
2. Summarize the results of Taylor's experiment.
3. To what extent do first grade children have out-of-school uses for arithmetic?
4. What operations did Miss Smith find to be most prominent in the first grade pupil's out-of-school uses of arithmetic?
5. Do you believe that opponents of arithmetic in the first grade may have been more annoyed by the kind of instruction given than by the fact that this subject was included in the curriculum? Discuss.

6. What kinds of number experiences do children usually receive in the kindergarten?

7. If Miss Harris' recommendations for the kindergarten are adopted, would you recommend that arithmetic instruction in the first grade be entirely "incidental"?

8. What are the dangers in "incidental" number instruction?

9. Why should the pupil's first experiences in addition be carefully supervised by a competent teacher? Discuss from the point of view of the psychology of habit formation.

10. Can you find any respects in which the recommendations of Miss Harris fail to agree with those of Miss Smith? How much measuring would each have done?

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Chapter III

COUNTING AND THE NUMBER CONCEPT

First grade teachers tell us that when children come to school they are usually able to count. Most of us have witnessed the pride of the six-year-old boy or girl who is permitted to demonstrate his ability to count to one hundred on the first day of school.

To say that a six-year-old child learns to count to one hundred before starting to school does not mean that this child understands the meaning of all two-digit numbers. In other words, this type of counting may precede the development of more than the most rudimentary sort of number concept.

Rote and rational counting. It is important that we distinguish between *rote* and *rational* counting. By rote counting we mean the mere naming of the number words—*one, two, three,* etc. These words may be no more meaningful to the child than any other jingle which he recites in his pre-school days. By rational counting we mean the actual identification of groups of objects or individuals through the use of number words; that is, the ability to associate *eight* with eight books, boys, pencils, etc.

An illustration. Several years ago the writer witnessed an incident which illustrates clearly the distinction between rote and rational counting and the fact that the former may exist without the latter. During the World War, while riding on a train he sat behind a little girl, perhaps five years old, and her father. The little girl was looking at a copy of the Literary Digest which had on the front cover page a picture of four soldiers in color, one each in the uniform of the four principal allied

powers. The following conversation took place between the girl and her father.

Girl: "See the men, Daddy?"
Father: (No response.)
Girl: "Do you want me to count them?"
Father: (Indifferently) "Yes, count them."
Girl: "One, two, three, four, five, six."
Father: (With more interest) "Oh no, that isn't right. Count them again."
Girl: "One, two, three, four, five, six, seven."
Father: "No, no, look at the picture and count them more carefully."
Girl: "One, two, three, four, five, six."
Father: "Now point to the pictures with your finger and count them again."
Girl: (Pointing) "One, two, three, four, five, six." (She pointed to each of the four soldiers separately but there seemed to be no relationship between her pointing and her counting.)
Father: "Now count them while *I* point to the pictures.
Girl: "One, two, three, four, five. (Evidently paying no attention to her father's pointing.)

The father recognized that his little daughter was utterly unable to count rationally. He thereupon proceeded to give her a lesson in counting. He insisted that she say *one,* while he pointed to the first picture, *two,* while he pointed to the second, *three,* while he pointed to the third, and *four,* while he pointed to the fourth, and that she stop counting when he stopped pointing. After several efforts she succeeded and was able to count the four soldiers correctly without her father's assistance.

Individual differences among pupils. It is important that the first grade teacher study the nature and extent of the individual differences among her pupils, especially as regards the extent to which they can count by rote and the extent to which they understand the meaning of number. All will probably be able to do some rote counting but there will be large differences due to differences in their pre-school play experiences and variations in the amount of coaching which they have received from parents and others at home. Some will be able to do rote counting to one hundred or beyond, while others may not know the

number words in their proper sequence to ten. Similar variations will be found in the extent to which they understand the meaning of number, and in the extent to which they can do rational counting. In the sixth year level of the Stanford Revision of the Binet-Simon intelligence tests, children are asked to count thirteen pennies. This means, according to the test data, that from two-thirds to three-fourths of unselected six-year-olds are able to do rational counting to thirteen.[1]

There is another respect in which first grade pupils will differ, *viz.*, in the understanding which they have of the meaning of number. Some will understand *six* to mean six objects or things (the cardinal meaning); others will think of six as *the sixth of a series* of objects or things (the *ordinal* meaning). It is extremely important that the first grade teacher find out just what numbers mean to each of her pupils, as well as the extent to which they can do *rote* and *rational* counting, before planning any incidental or other number instruction.

Cardinal meaning of number fundamental. The early plans of the first grade teacher should look definitely toward establishing the *cardinal* meaning of number. In other words, she should plan that her pupils think of *six* as *six things.* Later, the *ordinal* meaning may be presented through the use of the word, *sixth,* or the words, *sixth one.* Of course, the words *cardinal* and *ordinal* will not be used with the pupils.

The teacher should be careful in selecting and arranging objective materials to see to it that pupils get the cardinal meaning.

	1 11 111 1111		1 1 1 1	
For example		may be helpful but		will
	1 2 3 4		1 2 3 4	

probably make impressions which will have to be corrected, or at least modified in later lessons.

Sometimes teachers very definitely give pupils wrong initial impressions, or at least initial impressions which they do not in-

[1] Terman, Lewis M. *The Measurement of Intelligence.* Houghton Mifflin Company, 1916, pp. 48, 148, and 180-181.

tend to give them, through the procedures and materials which they use in teaching. For example, a first grade teacher who was teaching her pupils to read numbers (that is, the single digits) placed on the chalk tray of the blackboard cards on which were printed in large type the numerals, 1, 2, 3, 4, . . . 9. She would say, "Bring me the six, Harry," or, "Bring me four, Mary." *Four* came to mean, not four apples, pencils, children, or other objective materials, but a card on which the symbol "4" was printed. Obviously, "4" is not four, but is only a shorthand way of saying "four." First graders may quite properly read "4" as "four" provided that they fully understand that *four of something* is meant. Early number lessons, then, should be very definitely tied up with concrete, objective experiences.

The influence of Pestalozzi. The perceptual basis for number work is readily traced back to the work of that great educational reformer, Pestalozzi. One of Pestalozzi's most urgent bits of advice to teachers was "number before figure." To him there was a large difference between six sticks, six dots, etc., and the symbol, "6." He would, therefore, have pupils taught the meaning of number by counting things which they saw, heard, and touched—counting pupils, chairs, hand claps, bell sounds, beads, strings, etc. Pestalozzi went to an unfortunate extreme in having the meaning of number taught *exclusively* through counting a great variety of objective materials for each number taught, but to him must be given the credit for insisting that early number experiences must be concrete experiences with real objects.

Rote or rational counting first? Many writers have not only recognized that rote counting usually *does* precede rational counting, since children learn rote counting in their pre-school games and in imitation of others, but, furthermore, urge that rote counting *should* precede rational counting. They would have these children first learn to recite the number names—one, two, three, etc., and then later hope to attach definite meaning to each of these. This seems to be no more commendable than having pupils in the intermediate grades learn to spell geometric terms before

these terms are used in real situations, or having a pupil memorize a formal definition of "fraction" before he has had any experience with fractions. Ordinarily, it is better psychology to let the terms and definitions come along *after* some concrete experiences with the situations to which they apply, for they are then more meaningful and may clarify the pupil's thinking and round out his understanding of the subject. So, if we could have our way about it, we would have pupils learn the number names *while* learning to count real objects. But, if nearly all first grade pupils have already learned rote counting to *ten* or farther before entering school, the teacher's first concern should be to see that *all* acquire the ability to count real objects to the extent of at least twenty. For the few who have not yet learned the number names to ten, she may use the following rhymes which children seem to enjoy but which they will recite in more or less meaningless fashion. We recommend this procedure only because the pupils who have not yet learned rote counting are distinctly in the minority. If the teacher can so arrange her work that these few can be instructed to count by actually counting *things,* it will be distinctly better than the rote learning of rhymes.

One, two, three, four, five,
I caught a hare alive;
Six, seven, eight, nine, ten,
I let him go again.

One little, two little, three little Indians,
Four little, five little, six little Indians,
Seven little, eight little, nine little Indians,
Ten little Indian boys.

Use real and interesting objective materials. In making clear the meaning of numbers, a great variety of things should be counted and the results announced. Such questions as, How many rows of children have we? How many children in a row? How many boys in the first row? How many girls in the third row? How many buttons on William's sweater? How many wear glasses? etc., should be asked and answered. A game may be made of this work by having one pupil frame a question and call

upon another for the answer. If successful, he may ask the next question, etc.

Circles, dots, marks, pictures of apples and other pictured and printed materials may be used to an advantage, but such materials will probably be less interesting (they are certainly less real) than the objects, actions and sounds which the pupil encounters in his ordinary experiences. The informal and incidental connections which we can make with the acts and objects of everyday life experiences are more likely to be interesting and are certain to be better understood than are dots, lines, counters and other more or less formal materials.

Recognition of the size of groups. Pupils in the primary grades must at first count objects to determine the number in a group, no matter how small the group. Later a pupil will definitely recognize a group of three as containing three, without counting, that is, without saying, overtly or implicitly, "one, two, three." The proper development of number concept will include provision for training in the recognition of the size of small groups without counting. Flashing pictures of two, three, four, and five objects and calling for immediate responses as to the number will reveal the differences among pupils as to the extent to which they have developed this aspect of number concept and will be excellent practice. Doubtless some will always show marked deficiencies in this respect but it is important for the teacher to provide pupils with opportunities for practice in the ready recognition of the number in a small group.

The writer knows well-educated highly intelligent adults who are utterly unable to identify a group of four as *four* without counting. One, in particular, a university professor, must not only count the objects in such small groups but also touches them or points at them as he counts. It is an open question, to be sure, whether this condition is due to an innate deficiency or to inadequate training or to both, but pending an experimental study of the question, teachers should give pupils the kind of practice suggested.

Reading and writing numbers. In the first grade, while the pupil is being taught to count, and while he is enlarging his understanding of the meaning of number, he should frequently receive training in reading and writing numbers. He will first be taught to recognize the digits, 0 1 2 3 4 5 6 7 8 9, associating them with their names, zero, one, two, three, four, five, six, seven, eight, nine. Gradually, this work will be extended through the teens and the later decades until all two-digit numbers are included.

When the pupil is able to read and write all numbers to 20, it is well to emphasize the decade names by having him count to one hundred by tens and to read and write the corresponding numbers, 10 20 30 40 50 60 70 80 90 100. Then, the systematic arrangement of our number system and the similarity of the decades above 20 can be pointed out, and, with proper practice, readily learned.

The alert primary teacher will find many opportunities for practising reading numbers. Numbers on the calendar will afford practice as far as 31; pages in books usually run to 100 or higher; numbers are pasted on coat hooks where the children hang their wraps; house numbers run through two, three, four, and sometimes five digits; objects in shop windows are accompanied by price tags; the children play store, making and reading their own price labels; watches usually have the Arabic numerals, 1-12, as well as 10, 20, 30, 40, 50, 60 on the second-hand dial; telephone numbers will be looked up; automobiles may be identified by their license plates; etc. The large numbers found on license plates and in telephone directories may be read by first and second grade children, for we say, "six-two-four-seven-one-five," in reading 624-715 on a license plate, not "six hundred twenty-four thousand seven hundred fifteen." Later, of course, they will learn to read six-digit numbers in the more orthodox fashion.

It is imperative that the pupil receive close supervision in his early efforts to write numbers, particularly when learning to form the single numerals. If left to their own devices, these children

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will sometimes originate unique ways of making numerals, ways which are slow and difficult and which soon become well-fixed habits. The writer has frequently observed first grade children, in trying to make the figure 8 from copy set by the teacher, make two separate tiny loops, one over the other. Sometimes the loops failed to touch and sometimes they overlapped. In making the figure 7, they will sometimes make three separate marks, removing the pencil from the paper after each. They will originate startlingly novel ways of making 3's and may, in fact, get into difficulty with each of the numerals.

If the efforts of the primary teacher are well directed, pupils will learn to determine accurately the number of objects in a group by counting; will be able to compare groups, indicating which is the larger and which is the smaller; will learn to identify small groups without counting; will become proficient in reading numbers in situations requiring this ability; will form proper habits of writing numbers; and will gradually but continually enlarge their understanding of number meanings.

EXERCISES

1. Which is the more important, *rote* or *rational* counting? Why?
2. Should rote counting precede rational counting? Discuss.
3. What is a cardinal number? An ordinal number? Which of these two number meanings should come first in the pupil's experiences with number?
4. Give further suggestions as to how the teacher may inadvertently give pupils a wrong understanding of number meaning.
5. Of what value are rhymes in teaching counting?
6. If children are being taught to count objects, does it matter *what* objects they count?
7. How large a group can you identify without counting?
8. Test a few primary pupils to see how large groups they can identify at a glance, that is, without counting.

9. Have you known of cases of children who learned improper methods of forming digits or letters of the alphabet because of the lack of adequate supervision?

10. Make an inventory of the out-of-school needs of first grade pupils in your community for reading numbers. Of the school needs.

11. Make a similar inventory of the needs of second grade children, listing the activities involved and showing the size of the numbers in each activity.

12. Do you know of any further needs for reading numbers which third grade children possess?

13. What is meant by the last phrase of the chapter, "Gradually but continually enlarge their understanding of number meanings"?

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Chapter IV

TEACHING THE ADDITION AND SUBTRACTION FACTS

As indicated by the chapter heading, we have combined the teaching of the fundamental facts in addition with the teaching of the fundamental facts in subtraction. It seems wasteful to teach these processes separately. For example, having taught that 3 and 2 are 5, and that 2 and 3 are 5, it would seem to be only sensible economy in learning to proceed at once to teach that 5 less 2 are 3, and that 5 less 3 are 2.

The number of combinations. If we combine each of the nine digits with each of the remaining digits once, we shall have a total of 45 combinations. They are:

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
	2	2	2	2	2	2	2	2
	2	3	4	5	6	7	8	9
		3	3	3	3	3	3	3
		3	4	5	6	7	8	9
			4	4	4	4	4	4
			4	5	6	7	8	9
				5	5	5	5	5
				5	6	7	8	9

6 6 —	6 7 —	6 8 —	6 9 —
	7 7 —	7 8 —	7 9 —
		8 8 —	8 9 —
			9 9 —

But in the chapter heading, we refer to addition *facts*. Let us distinguish between an addition or subtraction *combination* and an addition or subtraction *fact*.

By an addition *combination*, we mean a combining of two digits in either of the two possible orders. That is, $2 + 4$ and $4 + 2$ are different ways of expressing the *same combination*. However, $2 + 4$ and $4 + 2$ are different facts. Thus, it will be seen that each addition combination, except the doubles ($1 + 1$, $2 + 2$, $3 + 3$, etc.), provides us with two addition facts. There are, then, 81 addition facts. In like manner, there are 81 subtraction facts. To get the 81 subtraction facts we list the 45 addition sums as minuends and in each of the 45 cases subtract the two addends separately. Thus, we have the addition combination, $6 + 4 = 10$. The two corresponding subtraction facts are $10 - 6 = 4$ and $10 - 4 = 6$.

The zero combinations. If we include in the list of addition combinations those involving zero, we shall have ten more addition combinations and 19 more addition facts, making a total of 55 addition combinations and 100 addition facts. The ten addition combinations are, of course,

0 0 —	1 0 —	2 0 —	3 0 —	4 0 —	5 0 —	6 0 —	7 0 —	8 0 —	9 0 —

These are ten of the 19 additional addition facts. The remaining nine are,

0	0	0	0	0	0	0	0	0
1	2	3	4	5	6	7	8	9
—	—	—	—	—	—	—	—	—

The 19 subtraction facts with zero are,

0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0
—	—	—	—	—	—	—	—	—	—
1	2	3	4	5	6	7	8	9	
1	2	3	4	5	6	7	8	9	
—	—	—	—	—	—	—	—	—	

We may sum up as follows:

Number of addition combinations, without zeros..	45
Number of addition combinations, with zeros	55
Number of addition facts, without zeros	81
Number of addition facts, with zeros...........	100

The number of subtraction combinations and facts in each group is the same as the number of addition combinations or facts in the corresponding group.

A question of some importance is, Should the zero combinations be taught along with the other addition combinations? Many teachers teach the zero combinations with the others, giving them the same emphasis which the others receive. The reason given is that tests show that pupils have trouble with the zero combinations. They will write $6 + 0 = 0$, or $6 + 0 = 7$, as well as $6 + 0 = 6$. Undoubtedly, very specific instruction must be given in the zero combinations in addition, but whether they should be taught as separate combinations is another question.

There are two considerations which would seem to justify the teacher in confining her early work in the addition combinations to the 45 combinations and 81 facts which do not include zero. In the first place, the zero combinations as separate combinations

do not occur in the pupil's real number experiences. He may have four cents and want to know how much he will have if he earns five cents more, thus having a real need for the combination, $4 + 5$, but if he has four cents he will not have occasion to wonder how much he would have if he should earn zero cents more. This latter type of problem, with its corresponding example, $4 + 0$, occurs only in the schoolroom.

In the second place, learning the zero combinations is not closely analogous to learning the other addition combinations. Each addition combination, involving two positive numbers, has a sum which is different from each of the addends and which must be learned and retained. But one can learn any number of zero combinations in one lesson, if his concept of number permits, by simply observing that any number plus zero gives the number we started with and that zero plus any number gives that number. Throughout his work in arithmetic and other mathematics, the pupil will learn that zero is unique and that the operations with zero are far from uniform. To scatter the zero combinations through the list of other addition combinations, making of each of them separate facts to be automatically mastered, means that we force upon the pupil more or less artificial and meaningless experiences, add considerably to his difficulties, and fail to take advantage of an attractive and easy way to master the zero difficulties in addition.

It seems that the proper place to teach the zero combinations is in connection with situations involving their use. Such situations arise in examples where a single addend has two or more digits, one of which may be zero, as $20 + 6$, $640 + 235$, or $325 + 104$. The addition of zero should be presented when such situations arise, and the two important principles—*any number plus zero equals the number* and *zero plus any number equals the number*—taught. This should be followed immediately by any abundance of practice with such examples as those given, to insure mastery.

For similar reasons, we would not teach the zero combinations

as separate combinations in subtraction. The following may be a very real problem for the youngster standing before the shop window, "If I buy a rope at the 9 cent sale, will I have enough left out of my 13 cents to get a five-cent top?" but such a problem as "Mother put 8 apples in the basket. Nobody ate any. How many apples were left in the basket?" is an arithmetical monstrosity and an absurdity to children. Likewise, the problem, "I had 2 cents and spent 2 cents for candy. How much did I have left?" is no problem at all to the pupil who is old enough and intelligent enough to appreciate its conditions. When children come out of the store with empty purses they do not compute to find out how much they have left. In subtraction, as in addition, there are two fundamental principles to be taught. They are: *any number less zero leaves that number,* and *any number less itself leaves zero.* These are not to be recited by the children, as is too often the case with rules, but are to be taught through the medium of real problems and examples.

The teacher's task in teaching the addition and subtraction facts is, then, the task of teaching 81 addition facts and 81 corresponding subtraction facts.

The order of teaching the combinations. We have indicated that there are 162 specific facts in addition and subtraction to be taught. Fortunately, the task is less formidable than it at first appears because we shall teach the addition and subtraction facts together and we shall present the two forms of a combination simultaneously. Any combination, then, except the doubles, suggests four facts, two in addition and two in subtraction. These four facts, organized into a lesson, we shall call a *teaching unit.* The combination 8 and 5, for example, suggests the teaching unit made up of the four facts, $8 + 5 = 13$, $5 + 8 = 13$, $13 - 8 = 5$, and $13 - 5 = 8$. In the case of the doubles, of course, a teaching unit includes but two facts, one in addition and one in subtraction. We have, for example, $6 + 6 = 12$, and $12 - 6 = 6$.

In what order should these 45 teaching units be presented to

the children? This is a question upon which there is considerable variation in practice. The older practice consisted of presenting these in the form of "tables," first all of the ones, then all of the twos, etc. This naturally led to counting as a means of arriving at a sum, for each combination so presented in a table was equal to just one more than the combination which preceded it. The present tendency is to present the combinations in a more or less miscellaneous order as a means of discouraging counting. In deciding upon an order for teaching these 45 teaching units, we have been guided by two principles which are of fundamental importance.

The two principles. First, the combinations should be presented in an order which will not encourage children to count. Counting is probably the greatest fault in addition which elementary school children possess. It may be natural that they should count to find a sum, particularly if the teacher's practices encourage them, but, that counting is a time-consuming and error-producing habit none will deny. We want children to learn to add with reasonable speed and with a high degree of accuracy. To accomplish this, counting must be discouraged in all addition experiences. The counting habit may be due to improper teaching methods or to the arrangement of the curriculum materials. We shall consider the former cause in later pages. As to the latter, it is important that no two consecutive combinations shall have a common addend with the other addends consecutive numbers, that $8 + 7$, for example, should not be taught immediately preceding or following $8 + 6$ or $8 + 8$ or $7 + 7$ or $9 + 7$.

In the second place, the combinations should be arranged approximately in the order of their difficulty. Just as we should present the easier words in spelling before the harder words, so we should take account of difficulty in determining the order in which number combinations should be taught. Several investigators have undertaken to determine the relative difficulty of the addition and subtraction combinations and, naturally, their results do not entirely agree. Probably the best of these studies

is that by Clapp.[1] He determined the difficulty of the combinations according to the number of errors made by 7,000 pupils in Grades IV to VIII, inclusive. He gives a list of the 100 addition combinations (zero combinations included) arranged in the order of their difficulty and another list of 100 subtraction combinations, similarly arranged. Naturally, the order in which the combinations appear in the two lists differs. But we can use Clapp's two lists in determining the order in which we shall present our 45 teaching units, thereby arranging them in approximately the order of their difficulty in both addition and subtraction.

Two groups of teaching units. For convenience in organizing the material we may divide the 45 teaching units into two groups. The first group will contain the 25 whose sums do not exceed 10; the second group will contain the remaining 20—those whose sums are greater than 10. This arrangement does not do great violence to the ranking according to difficulty. In fact, to get the 25 whose sums do not exceed 10, we need to include but 30 of the entire 45, when ranked according to difficulty. The five whose sums are greater than 10 (included in these 30) were moved up to positions in the second group.

Arranging the easier group of 25 in accordance with the two principles which have been stated, we have the following list.

2	1	1	3	1
2	1	4	3	6
—	—	—	—	—
4	1	1	5	1
4	2	7	5	3
—	—	—	—	—
1	2	1	4	2
5	3	8	5	4
—	—	—	—	—

[1] Clapp, Frank L. *The Number Combinations: Their Relative Difficulty and the Frequency of Their Appearance in Text-Books.* Bureau of Educational Research, Bulletin No. 2, Madison, Wisconsin: University of Wisconsin, 1924. 126 pp.

1	2	3	2	3
9	5	6	8	4
–	–	–	–	–
2	3	2	4	3
6	5	7	6	7
–	–	–	–	–

It is intended that the teacher should first present the combinations in the first line, then those in the second line, etc., until the twenty-fifth combination, $\begin{array}{c}3\\ \underline{7}\end{array}$, has been taught.

The second group, containing 20 harder teaching units, has been similarly arranged in accordance with the two principles, giving us the following list.

6	2	8	3	4
6	9	8	9	8
–	–	–	–	–
9	3	7	5	4
9	8	7	6	7
–	–	–	–	–
6	7	5	6	4
7	8	7	8	9
–	–	–	–	–
8	6	5	7	5
9	9	8	9	9
–	–	–	–	–

HOW TO TEACH THE COMBINATIONS

The counting habit to be avoided. We have said that the habit of counting in addition and subtraction is widespread among elementary school pupils. This fault can usually be traced to poor teaching in the first and second grades when the fundamentals of addition and subtraction were presented. If

primary teachers would but remember that the practices which they initiate are likely to become habitual practices lasting through the years of the elementary school and on through high school and life, they would doubtless be more concerned about the principles of correct habit formation. The writer has given addition tests to hundreds of college of education students—prospective teachers—and has found many whose papers showed unmistakable signs of counting; the columns were liberally besprinkled with pencil dots, indicating that the students had accompanied the counting process with pencil tapping. Counting not only consumes time but it is a prolific source of errors. It is extremely important, then, that early work in the fundamentals shall be so planned as to reduce this practice to a minimum.

Immediate automatic responses desired. Teaching the fundamental facts of arithmetic is a habit-forming process. What we desire to do is to form bonds in the child's nervous system between stimuli and responses. For example, when a child sees or hears, "5 + 9," we want him to react to this stimulus with the response, "14," and to respond immediately and directly, without following any roundabout procedure, such as saying aloud or to himself, "10, 11, 12, 13, 14."

Learning 45 addition combinations may be likened to the task of learning the names of 45 other children in one's room. The combinations are the individuals; the sums are the names. A pupil, coming into a strange school, experiences little difficulty and consumes little time in learning the names of 45 associates. Learning the 45 addition combinations is intrinsically no more difficult than learning the names of 45 strange people. Pupils are more successful with the latter because of a greater degree of interest in the situation, a better motive for learning. But when Harry sees his friend, William, a block away, his response, "Eh! Bill!" comes immediately and directly, and not by going through a series of intermediate responses, such as "boy, gray pants, blue sweater, cap to one side, coming from Elm Street, Bill." The details enumerated undoubtedly help in identifying

his friend, but they are seen simultaneously, come almost as a single stimulus and unconsciously lead to the response indicated.

Laws of habit formation. There are two important laws of habit formation—the *Law of Exercise* and the *Law of Effect.* The Law of Exercise states that bonds or mental connections which are used become strengthened while those which are not used become weakened. The Law of Effect says that if practice on a bond is accompanied or followed by pleasure or satisfaction, the bond becomes strengthened but that if dissatisfying or displeasing conditions accompany or follow this practice, the bond is weakened. According to the Law of Exercise, pupils must practise the fundamental bonds to insure their retention; according to the Law of Effect, this practice must take place under conditions which the pupils enjoy. The reason pupils are usually much more successful in learning the names of 45 classmates than in learning the 45 addition combinations is found largely in the Law of Effect.

The use of objects. Probably nearly all primary teachers will subscribe to the statements of the preceding paragraphs, admitting the desirability of ready responses unaccompanied by counting. But, on the other hand, many primary teachers, probably a majority, pave the way for the formation of the counting habit by the manner in which they use objects in teaching the fundamental facts of addition and subtraction. It is quite common for primary teachers to teach an addition fact, as $3 + 2$, in the following manner:

Teacher: (Placing two blocks on a table) "How many blocks have I here?"
Pupil: (After counting) "Two."
Teacher: (Placing three blocks in another group on the table) "And how many blocks have I here?"
Pupil: (Counting) "Three."
Teacher: (Pushing the two groups of blocks together) "How many have I altogether?"
Pupil: (After counting) "Five."
Teacher: "Then two blocks and three blocks are how many blocks?"
Pupil: "Five."

The teacher then repeats the lesson using some other kind of objects and then proceeds to provide abstract drill on the combination, with others which have been previously taught.

If unable to recall the sum of 2 and 3 the next day, the pupil, quite naturally, counts. He may reproduce for himself, the drama of the day before, counting groups of two and three objects of any kind available. Or, he may count by tapping his pencil, by touching his fingers to the desk, by tapping his foot on the floor, or his tongue against the roof of his mouth. He has not been taught that 2 and 3 make 5 but he *has* been taught how to find the sum of 2 and 3. With this information at hand, there is no motive for his retaining the sum in memory.

The tactics of the teacher instead of being an aid to learning are often an impediment. How many have witnessed such a scene as the following in a primary room:

Teacher: "How many are 9 and 6?" (Pause) "Percival."
Percival: (Silence).
Teacher: "Now, Percival, you know how many 9 and 6 are, I am sure. Think, Percival, think."
Percival: (After more silence) "Fifteen."
Teacher: "Fine! I was sure that you could tell me if you would just think."

After carefully training pupils to find sums in addition by counting, we spend much subsequent time trying to break up this habit. We teach combinations through the use of objects and then we suddenly remove the objects and tell the pupil that he must not count but must *think*. The pupil, denied the customary objects, substitutes others, as has been suggested, or *images* of objects. Counting the images is a more difficult process than counting the original objects, however, so errors are more frequent. "The images he gets in this thinking don't always stay at their proper place. Some slip away; some that are foreigners obtrude themselves. Some may come and some may go, but the child's struggle with errors goes on forever." [2]

[2] Myers, G. C. *The Prevention and Correction of Errors in Arithmetic.* The Plymouth Press, 1925, p. 10.

It is almost impossible to break up the counting habit when it has been established. The only sure cure is prevention. If objects are used they must be used in such a way as to facilitate direct responses, responses unaccompanied by counting.

In general, two extreme practices prevail in teaching these fundamental facts. There is the practice which induces counting, the practice which has been described. The other practice goes to the opposite extreme of teaching combinations as pure abstractions, independent of any concrete setting whatsoever. According to this practice, the pupil is simply told that 2 and 3 are 5 and that he must remember it. To make sure that he does remember it, a large amount of drill is provided, but the drill, dealing with a situation which is unreal to the child, is quite likely to be uninteresting. To stimulate interest the situation is sugar-coated by cleverly concealing the drill in games—games in which number wheels are drawn, bean bags are thrown, etc. The games may be good enough but the arithmetical facts which annoyingly obtrude themselves are made no less unreal and no less uninteresting thereby. Surely, a less objectionable and more desirable method than is represented by these extreme practices can be found.

There is a real value in the use of objects provided they can be used in such a manner as to make the meaning of the process clear and provided that the counting habit is not fostered thereby. In a former chapter we have recognized the value of objects in teaching the meaning of number and have seen that the influence of Pestalozzi lifted early lessons from the barren level of meaningless words to the plane of rich, concrete experiences. Surely, we do not wish to drift back to the pre-Pestalozzian period in the teaching of the fundamental operations.

In teaching an addition combination, as $5 + 4 = 9$, objects serve a dual purpose. First, they demonstrate to the pupil the truth of our statement that 5 and 4 make 9; second, they show the meaning of addition. These are very worthy purposes and are not to be treated lightly. The recognition of them is no doubt

responsible for the widespread use of objects in early work in the fundamental operations.

In the writer's opinion, the best scheme for using objects in teaching the addition and subtraction facts so as not to encourage counting is that which has been developed by Buckingham.[3] We shall illustrate his method by using the first combination in the list on page 33.

Teaching 2 and 2 are 4. "Let the teacher call to the front of the room four boys (or girls). The teacher first asks and secures an answer to the question, 'How many boys (or girls) are there?' At this point (*i.e.,* at the presentation of the *total* group) the pupils may count if they cannot give the correct answer without doing so. In this case, however, the pupils will almost certainly be able to reply correctly without counting.

"After securing several responses to the question, 'How many children are there?' the teacher may divide the four children into groups of two, keeping the groups close together.

"Again let the teacher ask: 'How many children did you say there were?' or 'How many children are there altogether?' The answer should be obtained and repeated if there appears to be any doubt about it.

"Teacher: 'How many children are there here?' indicating one of the groups. Pupil: 'Two.' Teacher: 'How many are there here?' indicating the other group. Pupil: 'Two.' Teacher: 'How many children are there altogether?' Pupil: 'Four.' Teacher: 'Then two children and two children are how many children?' Pupil: 'Four.'

"The teacher may then apply the same method to other concrete material, such as books, pencils, splints, beads, etc. But in all such cases the counting, if there is any, should be *before* the whole group is separated into two parts."

The outstanding feature of Buckingham's plan is that it employs the simple, yet very important device of *presenting the*

[3] Buckingham, B. R., and Osburn, W. J. *The Buckingham-Osburn Searchlight Arithmetics, Introductory Book.* Ginn and Company, 1927, pp. 28-29.

whole group before the component parts and of confining the pupil's counting of the whole group to the time of the initial presentation. Right here lies the weakness of the usual plan of employing objects. The pupil counts the two groups separately and then counts the total. Is there any wonder that he develops the habit of counting to find the sum of two numbers? Buckingham's plan seems to possess the happy feature of getting the benefit of concrete experiences which objects afford, without inducing the habit of counting.

This, then, is the plan which we recommend for teaching each of the 45 combinations in addition and subtraction, the 45 teaching units. Use objects to make the experience concrete and real, but use objects in such a way that counting will not suggest itself to the pupil.

After the combination has been presented, according to the plan described above, using a variety of objective materials, it should be written on the blackboard and written by the pupils at their seats. The teacher should write $\begin{array}{r}2\\ \underline{2}\\ 4\end{array}$, saying as she writes, "Two and two are four." The pupils should be told to say to themselves, "Two and two are four," as they write and should say and write this combination several times to insure its retention in memory.

Teaching the subtraction fact. After having presented the addition fact, $2 + 2 = 4$, the corresponding subtraction fact, $4 - 2 = 2$, should be presented. Again, objects may be used. The teacher may use pencils this time. Holding four pencils up where the entire class can see them, the teacher will ask, "How many pencils have I here?" Pupil: "Four." Teacher: "If I take away two pencils (removing them with her other hand and putting them out of sight) how many have I left?" Pupil: "Two." Teacher: "Then four pencils less two pencils are how many pencils?" or, "Then four pencils *take away* two pencils leave how many pencils?" Pupil: "Two." The latter form of question is less elegant but perhaps better understood by the pupil in his

first experience with subtraction, since the language fits better the teacher's actions. The English is improved if put in the following form: If from four pencils I take away two pencils, how many pencils have I left?

Present each of the 45 teaching units in the manner described, using objective materials for both the addition and the subtraction combinations. Remember, that in the case of 36 of the 45 teaching units, all but the doubles, each teaching unit means four facts to be taught. In the case of the $\begin{matrix}3\\ \underline{5}\end{matrix}$ unit, for example, we shall teach that 3 and 5 are 8, and immediately thereafter, that 5 and 3 are 8, and then proceed to the corresponding subtraction facts, 8 take away 5 are 3, and 8 take away 3 are 5. The pupil will write each of these four facts equally often, saying them to himself as he writes them.

Providing for continued practice. When a teaching unit is first presented, it should be presented in rather elaborate and painstaking fashion. Immediately after this initial presentation, the pupil should have considerable practice in writing the four (or two) forms. Never give more than one teaching unit in a lesson. Go slowly and be thorough if you would have the pupils remember the facts which they have been taught.

It is very essential, however, that careful plans be laid for continued practice on each of these fundamental bonds. Psychologists tell us that the best method of distributing the practice with a bond or group of bonds is to give at the time of first learning enough practice to form the bond rather well, and then to give practice in smaller and smaller amounts at longer and longer intervals. Some time after a teaching unit is presented, practice on the facts of that unit may be provided at long intervals only, but immediately following the teaching of a combination, practice should be frequent.

Resourceful teachers have developed many excellent games and devices for making this continued practice more enjoyable. A few of these are described in later pages and many more will be

found in the references listed at the end of this chapter. But whatever other drill devices one may employ, properly prepared flash cards are indispensable. There should be a card for each of the 162 addition and subtraction facts. The teacher may easily make them by hand, using India ink and a small camel's hair brush. She need not make the entire 162 at one time but can make them as needed—four for each teaching unit (two for the doubles). The figures should, of course, be large enough for pupils in any part of the room to read them. Each card should be printed on both sides—one side without the answer and the other side with the answer. For example, a card may show $\begin{array}{r}2\\ \underline{5}\end{array}$ on one side and $\begin{array}{r}2\\ \underline{5}\\ 7\end{array}$ on the other side.

One hundred sixty-two cards will be a cumbersome bunch to handle, if all grouped together. To make them easier to handle and to make it easier to find a particular combination, the 162 cards should be divided into four groups. Two of these groups will contain the 81 addition facts and two others, the 81 subtraction facts. Each of these groups may then be divided into two smaller groups, putting into one those whose sums or minuends do not exceed 10 and into the other those whose sums or minuends do exceed 10. There will then be 45 cards in each of the groups whose sums or minuends do not exceed 10 and 36 cards in each of the groups whose sums or minuends do exceed 10.

If counting is to be successfully prevented none but immediate responses should be accepted in a drill lesson. The reason for having the combination with the answer on the back of the card will now be apparent. When a card is "flashed," that is, when a combination is exposed, the pupil should respond at once with the sum or difference. If he hesitates, turn the card so that he can see the same combination with the answer and have him read the answer. In conducting a drill lesson with flash cards the name of the pupil who is to respond should be called as the card

is flashed and not after the card has been exposed, so that the time required for his response may be reduced to the smallest amount possible. Of course, if a pupil fails to respond, or responds incorrectly, and is shown the correct answer, he should be called on again later in the same lesson for the number fact on which he failed.

In conducting practice lessons on the fundamental number facts, the important thing to keep in mind is that the pupil must not be given time enough to count, or follow some other roundabout procedure in finding the answer. If he does not know the answer he should be told it, and told it promptly and effectively. Never *urge* pupils to respond in such drills if they hesitate. Pupils should not guess, should not take chances in this work. If the initial presentation is carefully planned and thorough and if the practice periods are sufficiently numerous and interesting, there is little likelihood that the facts will be forgotten. Permanent retention and immediate response is the only acceptable standard for this work.

It is a good plan to post these number facts *with the answers* in a conspicuous place immediately after they are taught and to leave them posted until there is no doubt that all members of the class know them. They may be written in large, plain figures on a section of the blackboard reserved for this purpose or the flash cards may be hung where all can see them. When a pupil feels in doubt about a combination he should not be criticized for looking at the posted list but should be encouraged to do so. This is an important point in early work in habit-forming subjects. The danger is that, soon after a number fact has been presented, a pupil, in attempting to recall it, may get the wrong answer and repeat that wrong answer sufficiently often to habituate himself in error. Myers gives interesting examples of the wrong responses of pupils to certain combinations. He shows that a particular wrong answer occurred again and again even though it was corrected each time it appeared, indicating that this wrong

impression had been made early in the pupil's experience with that combination.[4]

Instead of posting the combinations, some teachers provide the pupils with individual mimeographed keys. That is, each pupil is supplied with a sheet of paper on which are mimeographed the combinations with their sums, or differences. The individual keys have certain advantages over the plan of posting combinations. They prevent the eye strain which may be involved in looking at the combinations on a distant section of the blackboard or on cards hung in a remote position. On the other hand, if the keys are used, we shall either have to give the pupil a key as soon as we begin to teach the combinations and thus give him combinations which have not yet been taught, or we must wait until a fairly large group of combinations has been taught—enough to justify making a key—and then furnish them in this form. It seems best to post the combinations until the first group of 25 teaching units has been taught and then furnish these to the pupils in the form of a key if they seem to be needed. After the remaining 20 teaching units have been presented, they can be incorporated in a second key.

We may some day discover, however, that the best way to teach the primary facts of arithmetic is to give each pupil a complete key at the outset and let him begin immediately to solve simple problems and examples in which the different facts are used, allowing him to learn these facts gradually through use. Much of our learning is done in this way. A teacher with a new group of pupils frequently makes a chart of the room, writing the names of the pupils in small rectangles corresponding to the positions of their seats in the room. With the chart before her she soon learns to call all of the children by name. Sufficient experimental evidence has not yet been provided, however, to warrant the general use of this method in teaching the primary facts of arithmetic.

[4] Myers, G. C. *Op. cit.*, pp. 5-7.

If a key is used, the facts should be set down in systematic order so that any fact can be found without difficulty. The 81 addition facts may be made into a key as shown on the following page. If two keys are made—one containing the 45 facts whose sums do not exceed 10 and the other the 36 facts whose sums exceed 10—those at the left and above the stairstep line will be included in the first key and those at the right and below, in the second. The arrangement of the items in each key will be in the form of a triangle. In the former, the base will be at the top; in the latter, at the bottom.

To make the subtraction key, use the sums of the 81 addition facts as minuends and the first of the addends as subtrahends. The first 18 of the 81 facts in the subtraction key will then appear as follows:

2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1
—	—	—	—	—	—	—	—	—
1	2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10	11
2	2	2	2	2	2	2	2	2
—	—	—	—	—	—	—	—	—
1	2	3	4	5	6	7	8	9

For reasons already given we have omitted the zero facts. These will be taught later.

GAMES AND DRILL DEVICES

The teacher should have at hand a large number and variety of games to be used in relieving the monotony of drill lessons. In selecting games one needs to be careful that he does not choose games which induce counting or otherwise form bad habits in number work. Furthermore, the game should give the needed practice to as many of the class as possible, preferably to the

1 1 — 2	1 2 — 3	1 3 — 4	1 4 — 5	1 5 — 6	1 6 — 7	1 7 — 8	1 8 — 9	1 9 — 10
2 1 — 3	2 2 — 4	2 3 — 5	2 4 — 6	2 5 — 7	2 6 — 8	2 7 — 9	2 8 — 10	2 9 — 11
3 1 — 4	3 2 — 5	3 3 — 6	3 4 — 7	3 5 — 8	3 6 — 9	3 7 — 10	3 8 — 11	3 9 — 12
4 1 — 5	4 2 — 6	4 3 — 7	4 4 — 8	4 5 — 9	4 6 — 10	4 7 — 11	4 8 — 12	4 9 — 13
5 1 — 6	5 2 — 7	5 3 — 8	5 4 — 9	5 5 — 10	5 6 — 11	5 7 — 12	5 8 — 13	5 9 — 14
6 1 — 7	6 2 — 8	6 3 — 9	6 4 — 10	6 5 — 11	6 6 — 12	6 7 — 13	6 8 — 14	6 9 — 15
7 1 — 8	7 2 — 9	7 3 — 10	7 4 — 11	7 5 — 12	7 6 — 13	7 7 — 14	7 8 — 15	7 9 — 16
8 1 — 9	8 2 — 10	8 3 — 11	8 4 — 12	8 5 — 13	8 6 — 14	8 7 — 15	8 8 — 16	8 9 — 17
9 1 — 10	9 2 — 11	9 3 — 12	9 4 — 13	9 5 — 14	9 6 — 15	9 7 — 16	9 8 — 17	9 9 — 18

entire class simultaneously. Consider, for example, the following game.

Blind Man

> The children form a circle leaving one child in the center blindfolded. Each child has a card hung from his neck containing a number not greater than 9. The blind man catches someone. The one caught gives the sum of his number and that of the blind man. The blind man then guesses the number of the one caught. If he guesses the right number, the one caught becomes the blind man.

This game requires addition and subtraction. But as a drill device in addition it is very poor, for each pupil has but one combination—the sum of his own number and that of the blind man—and he knows in advance what that combination is and has an abundance of time to find the answer by counting. As a device for giving training in the subtraction combinations, it is but little better. The sole recipient of the benefits of the drill is the "blind man" and he has but one combination to give. This is a very small return for the amount of time consumed by the game. Furthermore, the author of the game states that the "blind man" *guesses* the number of the one caught. This, in the light of the requirements of a good drill lesson, is intolerable.

Here is another game which defeats its own purpose.

> The teacher writes on the blackboard: 1 2 3 4 5 6 7 8 9. She also has a large card with a number, as 2, written on it. She holds up the card and says, "Who will help me make five?" All the children stand who can tell what with 2 makes 5. The teacher then names some child. The child named goes to the board and points to 3 saying, "I am 3. I will help you make 5."

Those familiar with primary rooms in which this sort of thing is practised can visualize a group of small children looking at the 2 and touching the tips of the fingers of the left hand with the index finger of the right hand and whispering while they count

"three, four, five." Indeed, this game is most excellent training for the kind of thing which good primary teachers strive to avoid.

One more example of the kind of game which teachers of primary arithmetic should avoid will be given.

Bursting Balloons

> Draw a number of balloons upon the board, bringing all the strings together at the bottom. Put a combination on each balloon. When a child gives the correct answer to a combination he "bursts" the balloon and it is erased. Continue until all of the balloons are gone.

The fault of this game is clearly apparent. It does not require immediate responses. It provides a most excellent opportunity for counting.

We repeat that if games are to provide valuable practice on the fundamental number combinations, they must provide that practice under proper learning conditions; that is, they must reduce the opportunities for counting to a minimum and they must provide for the participation of a large number of pupils. They must, of course, be games which children like to play.

Examples of good games and drill devices. The greatest benefits probably come from flash card drills and written speed tests covering the combinations which have been taught. These, however, will become monotonous if used continually. Games offer the needed variety and are usually more interesting. The following game is very popular and may give excellent practice.

Pussy-in-the-corner

> All the pupils, except one who is "it," are arranged in a circle. Each pupil in the circle is given a number, not higher than 18, the same number being given to two pupils. The pupil who is "it" takes his place in the center and announces a combination, as 8 and 6. The two pupils who have the number 14 exchange places and the pupil who is "it" tries to get the place of one of

> them. If he is successful, the one displaced becomes "it." If he is unsuccessful, he announces another combination.

This game requires close attention and alertness from all pupils. One must recognize immediately any combination whose sum makes his number. If the pupils' numbers are changed frequently, each receives practice on a wider variety of combinations.

This game may also be used for subtraction combinations, provided that none of the numbers assigned is greater than 9. The one who is "it" then says, "14 take away 6," and those having the number 8 exchange places.

Fox and Geese

> The fox stands in the center of a circle of geese. He calls on a goose by name and announces a combination. If the sum is not given correctly, the goose is caught and joins the fox. Then another goose is named and a combination is called. A goose who has been caught may escape to the circle by giving a correct answer when the fox has accepted a wrong answer or he may be allowed to return to the circle by giving any answer before the goose called upon can give it, in which case the goose called upon is caught.

This game can be used equally well for the subtraction combinations. Always have the fox call a goose by name *before* announcing the combination.

Racing games. Many games require a pupil to speed over a list of combinations, trying to do them in fewer seconds than those who preceded him, trying to lower his own record of the day before, or trying to beat another pupil who is doing the same list or a list of equal difficulty. If two or more pupils race simultaneously, the answers should be written for in the confusion of two or more giving oral answers, it is difficult for the teacher and other pupils to detect mistakes.

Combinations may be arranged in a column or placed on a ladder as illustrated. The illustration shows two ladders each

containing 15 combinations. The combinations have been picked from those whose sums do not exceed 10. Note that 13 of the combinations given in the two ladders are the same but are expressed in different order. In half of these, the one ladder has the larger number stated first while in the remaining half the other ladder has the larger number stated first. Two of the combinations in each ladder are doubles not differing very greatly in difficulty. The ladders may be drawn on the blackboard and two pupils told to race to the top being careful not to fall (giving wrong answers). The answers are written along the right-hand side of the ladder. Pupils may be timed with a watch having a second hand, two seconds being added to each record for each mistake made. The pupil with the lowest net score wins.

4 + 2	2 + 4
6 + 3	3 + 6
3 + 7	7 + 3
4 + 1	1 + 4
6 + 4	4 + 6
2 + 3	3 + 2
1 + 9	9 + 1
1 + 1	2 + 2
5 + 5	3 + 3
3 + 5	5 + 3
7 + 2	2 + 7
4 + 3	3 + 4
2 + 6	6 + 2
1 + 8	8 + 1

If the equations form of statement has not been learned—and this form should probably not be taught until the pupils have learned the combinations stated in the usual vertical form—the combinations may be written in a row instead of a column and the races carried out as before.

Each teacher can make up a large number of sets of such ladders for the combinations whose sums do not exceed 10, for those whose sums do exceed 10, and for the two groups combined. These sets of ladders can be saved for use with later classes. Subtraction combinations can, of course, be practised with similar devices. Care should be taken to see that each combination appears occasionally and that the combinations which are most often missed on tests be given special emphasis. If one pupil

only is "climbing the ladder" he can give his sums orally, the teacher timing him and the other pupils watching for mistakes.

Stepping Stones

> Draw a stream with stepping stones so arranged that there are several different routes from one shore to the other. Each stone will have a combination written on it. Children choose routes and cross the stream by answering the combinations on the stones, being cautioned not to fall into the water by making a mistake. This game can be used for racing also. Children at their seats watch for mistakes.

This game may be varied by having the children choose up, forming two sides. Have as many as ten or a dozen routes across the stream. Two pupils, one from each side, go to the blackboard, each side sending its racers in the order in which they were chosen. Each child picks the route which his opponent shall take, the routes being of about equal difficulty. The two start at a signal from the teacher and race across. A point is scored for the side whose racer makes the lower net record, two seconds being added for each mistake made.

A few additional games will be given to illustrate more fully the types which should be used. Hundreds of games may be found in the references listed at the close of this chapter. Our purpose here is not to supply a long list of good games but to give a few samples illustrating the qualities of good games, so that teachers may intelligently choose others from any available sources.

THE GUESSING GAME

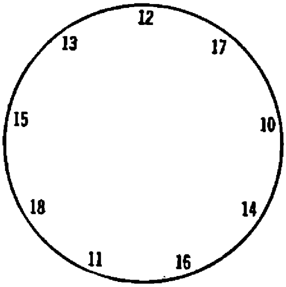
Teacher: "I am thinking of two numbers which make 12. Ruth."
Ruth: "Are you thinking of 8 and 4?"
Teacher: "No, I am not thinking of 8 and 4. Mildred."
Mildred: "Are you thinking of 7 and 6?"
Teacher: "No, Mildred, 7 and 6 make 13. I am thinking of two numbers which make 12. Betty."
Betty: "Are you thinking of 7 and 5?"
Teacher: "Yes, I am thinking of 7 and 5."
Betty then takes the teacher's place and the game proceeds.

Number Basket Upset

This game resembles the old-fashioned game, Fruit Basket Upset. The children sit in a circle with one standing in the center. Each child in the circle has a number pinned on him. The child in the center gives a combination three times in quick succession, as "6 and 4, 6 and 4, 6 and 4." The child whose number is 10 must say "10" before the one in the center has finished or be "it." At any time the child who is "it" may say, "Number basket upset," whereupon the children all exchange seats, the one in the center trying to get a seat. The one left standing is now "it" and the game proceeds.

Hurdle Race

Set down ten or a dozen addition combinations (or



6	4	9	8	7
8	7	6	8	5
—	—	—	—	—
9	8	3	9	6
4	9	7	9	6
—	—	—	—	—

subtraction combinations) and draw a large circle, writing just inside the circumference of the circle the sums (or differences) of the combinations as shown. Divide the class into two equal groups, or have two leaders choose sides. Choose a pupil from each team for a race.

The teacher points to a combination and the two pupils race to see who can first touch the correct sum in the circle. The pupil winning jumps the hurdle and scores a point for his side. The same two children may race for four or five combinations and be replaced by two new racers.

Drum Corps

A list of combinations is placed on the board. The class is divided into two groups taking the combinations alternately. If the pupil called upon gives the correct answer, the entire class drums on the desks with closed fists and a point is scored for that side. Someone is then called upon from the other side in the same manner. If an answer is wrong and someone on either side drums, a point is deducted from the score of his side.

SOLVING PROBLEMS

In a later chapter we shall consider problem solving in some detail, pointing out the characteristics of good problems and offering suggestions for training children to solve problems. At this point, we merely wish to emphasize the fact that problems—verbal problems—should be presented quite early in a child's arithmetical experiences. When a pupil has learned the first five teaching units, given on page 33, he should be given a number of problems whose solution calls for the skills which have been acquired. Indeed, problems will be given *in connection with* the teaching of any addition or subtraction fact. As soon as "2 and 2 are 4" has been objectively represented, problems like the following may be proposed: (*a*) Eleanor spent two cents for a valentine and two cents to send it to her cousin who lived in another city. How much did Eleanor spend altogether? (*b*) Mrs. Dixon bought four lemons and used two of them to make some lemonade. How many lemons did she have left? This will add interest to the work, if the problems are well selected, and will show the pupil the usefulness of the skills which he has been striving to acquire.

The first problems will, of course, be one-step problems, in-

volving the addition and subtraction facts which have been taught to date. A group of problems will be given after the first five teaching units have been presented, another group following the second five, etc. In the second group of problems will be found a few reviewing the facts of the first group of five teaching units, in the third group of problems the facts of the first ten teaching units will be reviewed, etc. As the combinations become better and better learned, they will receive less emphasis in problems, as well as in games and formal drill lessons.

In selecting and formulating problems, each of the 162 addition and subtraction facts should receive specific attention. In other words, proper attention to all these facts will require, in the course of the months used in learning the 45 teaching units, hundreds of problems. To illustrate this point and to suggest the kinds of problems which young children enjoy, we shall give a list of problems based upon the first five teaching units (page 33). Since three of these teaching units are doubles, there are 14 different addition and subtraction facts involved, so our list must contain at least 14 problems. It would be well to supply 28 or more, two or more for each of these facts. To economize space we give only the minimum list of 14.

1. When Lucile was visiting her grandmother, who lived on a farm, she went hunting for eggs. She found two eggs in the haymow and two more under the corn crib. How many eggs did Lucile find?
2. Lucile's grandmother was glad to get these four eggs. She used two of them in baking a cake. How many eggs did she have left?
3. The next day, Lucile went to look for eggs again. This time she found just one egg in the haymow and one under the corn crib. How many eggs did Lucile find this time?
4. On the way to the house with her two eggs, Lucile dropped one and broke it. How many did she have left to give her grandmother?

5. Mildred had only one picture book but Santa Claus brought her four more. How many picture books did Mildred have then?
6. Four mice were playing on the barn floor. Then one more mouse came out. How many mice were then playing on the barn floor?
7. Harry had five pieces of candy in his pocket. He ate one. How many pieces of candy did Harry have left?
8. Five blackbirds were sitting in the tree. Four flew away. How many blackbirds were left sitting in the tree?
9. One bright spring day Thomas went fishing. He caught three fish down by the big elm tree and three more under the bridge. How many fish did Thomas catch that day?
10. Thomas was very proud when he got home with his six fish. His mother said: "Three of these are too small to cook, but I shall cook the others. "How many fish did Thomas' mother cook?
11. Mother had only one orange but the grocery boy brought her six more. How many oranges did mother have then?
12. Jane had six dolls and her aunt gave her one more for Christmas. How many dolls did Jane have then?
13. John had seven cents and spent six cents for a big top. How many cents did John have left?
14. There were seven cookies in the jar. When William came home from school, he ate one. How many cookies were left in the jar?

In selecting and formulating problems, try to have them deal with situations in which children are interested and try to word them in such a way as to make a strong appeal to their imaginations. As to this matter of wording problems we shall have more to say later. But note the stronger appeal which problem 1 makes compared to the appeal which it would make were it worded in the usual matter-of-fact fashion, "Lucile found two eggs in one nest and two eggs in another nest. How many did

she find in all?" or, worse still, "There are two eggs in one nest and two in another. How many in both nests?"

TESTS ON THE FUNDAMENTAL FACTS

There should be frequent written tests on the facts which have been taught. It is best to mimeograph the tests, supply a copy to each pupil, have them start together, and allow sufficient time for the deliberate pupils to write the answers but not enough time for them to secure the answers by counting. It would be well to have a test on the fourteen facts of the first five teaching units when these have been presented. The following addition and subtraction facts would be included. Note that they are not listed in the order in which they were taught.

1	4	6	2	2	5	1
4	—2	1	—1	2	—4	6
–	–	–	–	–	–	–
6	7	4	7	5	3	1
—3	—1	1	—6	—1	3	1
–	–	–	–	–	–	–

Another test may be given when the second five teaching units have been taught. This test should include the above 14 facts and the 16 facts of the second group, 30 in all. In setting them down they may be arranged in random order and the 14 facts of the first group mixed with the 16 facts of the second group. The following is one of the large number of possible arrangements.

4	3	6	3	3	4	3	5	2	5
—1	—2	—3	—1	3	—3	1	—1	1	—4
–	–	–	–	–	–	–	–	–	–
1	10	1	8	4	5	1	8	4	7
2	—5	3	—4	1	5	4	—7	4	—1
–	–	–	–	–	–	–	–	–	–
2	1	7	8	7	4	6	2	1	1
—1	1	—6	—1	1	—2	1	2	7	6
–	–	–	–	–	–	–	–	–	–

The third group of five teaching units, $\underline{\begin{matrix}1\\5\end{matrix}}$ $\underline{\begin{matrix}2\\3\end{matrix}}$ $\underline{\begin{matrix}1\\8\end{matrix}}$ $\underline{\begin{matrix}4\\5\end{matrix}}$ $\underline{\begin{matrix}2\\4\end{matrix}}$, contains no doubles so there are 20 facts to be learned and tested, making a total of 50 so far. The test may include the entire 50 if the teacher thinks best but there will probably be some facts in the first group, and possibly some in the second, which can be safely omitted from the test. The items can easily be arranged in the manner indicated for the first and second tests. Those to be omitted, if any, will be selected in the light of the results of the first and second tests. Omitted items should occasionally be included in later tests to make sure that they have not been forgotten.

Recording the results. To get the maximum benefit from a test, the results should be tabulated and studied with care. A sheet of paper may be ruled into as many columns as there are items in the test and as many rows as there are pupils. Tally the wrong answers by putting an "X" underneath the item missed and opposite the name of the pupil missing it. Totals along the right-hand side of the sheet will show the number of examples missed by each pupil, thus identifying those in need of special instruction. Totals at the bottom of the sheet will tell the number of times each example was missed, showing which facts are most in need of further drill. The following is a portion of such a record sheet for the 14 items of the first test:

Names of Pupils	1 4	4 2	6 1	2 1	2 2	5 4	1 6	6 3	7 1	4 1	7 6	5 1	3 3	1 1	Total
Clara			X								X				2
Mabel															0
John		X				X		X			X				4
William											X				1
Richard															0

The record shows that John has missed four of the 14 examples, each of which is a subtraction example. Evidently something has gone wrong with his learning of the subtraction facts. Frequently one can locate the source of trouble in such a case by sitting down with the pupil and having him say aloud what he thinks as he does the examples of the test. Many other valuable bits of information can be gleaned from such a record sheet, such as the fact that 7 — 6 is a frequent source of trouble. This kind of detailed analysis will much more than pay for the time and energy required to make it and will pay in the best of dividends. Only through such a detailed diagnosis of individual difficulties can an effective program of remedial instruction be planned.

As indicated in Chapter II, practices differ greatly as to the amount of arithmetic which shall be taught in each of the primary grades. Some would have the work which has been discussed in this chapter taught in the first grade, others would postpone it until grade two is reached, some would offer a portion of it—say, the 25 easier teaching units—in grade one and the remainder in grade two, and a few would offer none of it before grade three. Whatever one may recommend, then, is certain to be acceptable to only a portion of the primary teachers and supervisors of this country. It is the opinion of the writer that all of the fundamental facts in addition and subtraction *can* be successfully taught in the latter half of the first year, but whether they *should* be is another question. The principal business of the first grade teacher is to teach children to read and nothing should be permitted to get in the way of this program. If a portion of the addition and subtraction facts can be taught in the first year without interference with the pupil's progress in reading, there seems to be no good reason why this should not be done, for children's out-of-school activities and interests certainly justify the procedure; but if this work is entirely postponed until the second year, there will probably be little, if any, ultimate loss, provided that the pupils do not develop counting habits to find sums. First grade teachers should be so well informed on the

out-of-school number experiences of their pupils that they can step in with a program of instruction in the fundamental facts of addition and subtraction if evidences of the counting habit manifest themselves.

Teachers should not get the impression from this chapter that the 45 teaching units in addition and subtraction are to be presented as a continuous program. To prevent monotony and loss of interest, other types of number work will occasionally intervene. The pupils should have practice in reading and writing numbers, say, any two-digit number in the first grade and any three-digit number in the second grade; in counting by 1's, 2's, 5's, and 10's, and in grouping objects into 2's, 3's, 4's, 5's, 10's, etc.; in getting acquainted with the signs, +, —, =; in learning to recognize coins; in reading Roman numbers to XII; in using a foot ruler for measuring; in learning to estimate distances; etc. But when one has launched into these teaching units, as many as five or ten should be presented before other number work intervenes. Furthermore, there should be no long intervals without practice on the facts which have been taught, or they may be forgotten.

The whole purpose is to present these facts with a maximum of interest and to distribute the practice so that they will be permanently retained.

EXERCISES

1. As used in this chapter, what do we mean by the terms, "addition combination," "addition fact," and "teaching unit"?
2. We gave a list of 45 teaching units, 45 addition combinations, and 81 addition facts. How many would we have in each of these three groups if the zero combinations were included?
3. Summarize the argument for excluding the zero combinations and treating them as a group later.
4. What considerations should determine the order in which the teaching units are presented?

5. Why should we object to having pupils count to find sums or differences?

6. What is the bearing of the laws of habit formation on the teaching of the fundamental facts of addition and subtraction?

7. In the references at the end of this chapter, are recommendations for the use of objects made in such a manner as to avoid engendering counting habits?

8. Observe the teaching of the fundamental combinations in addition and subtraction in two or three school rooms and critisize what you see, in the light of the discussions of this chapter.

9. What are the outstanding features of Buckingham's plan for teaching the combinations?

10. Is speed a remedy for counting?

11. What principles should govern the distribution of practice on a fundamental bond after it has been presented?

12. Summarize the directions for conducting flash card drills.

13. Describe two number games which are faulty, pointing out the faults, and suggesting means for improvement.

14. Describe two games which you consider well adapted to drill in the addition facts; two which you would recommend for the subtraction facts.

15. Write four problems, which you think would appeal to pupils, for the combination, $\begin{array}{r}4\\ \underline{6}\\ 10\end{array}$.

16. Prepare a test covering the first 15 teaching units, as given on page 33. Have each fact appear once in the test.

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CHAPTER V

THE TEACHING OF ADDITION

If the recommendations of the previous chapter have been carefully followed, the pupils should have ready control over the primary facts of addition and subtraction. It will be recalled that the aim was to give the pupil an automatic mastery of these facts. The dangers of counting were very specifically pointed out and no roundabout methods of finding sums were endorsed.

Our goal, in other words, was the setting up of *bonds* or *neural connections*. For example, we want $6 + 8$ to be so definitely and effectively connected with 14 in the pupil's nervous system that when he sees or hears $6 + 8$, the response, "14," will be forthcoming without delay. Some authors recommend that the combinations involving 9 be taught by having the pupil note that the sum is in the "teens" and is one less than the digit added to 9, and, similarly, that two less than the number added to 8 shows the number of "teens" in the sum. Just such practices as this are responsible for the time-consuming, error-producing methods which pupils often follow.

Analyses of the procedures which children follow in adding show clearly that in a large proportion of the cases these procedures are complex and not the straightforward action of simple bonds. Some add up and others add down; some count to find sums as they go up or down the column and others add each number to the preceding sum directly; some seek out combinations which make 10, 9, 5, etc., while others take each number as it comes; some split up the larger digits, adding 8 as 5 and 3, etc., but others add all digits, small and large, directly; some add

a large digit by adding 10 and subtracting the difference between the digit to be added and 10 (as $28 + 8 = 38 - 2$) but others seem not to have discovered this procedure; some add two numbers differing by one by doubling one number and adding or subtracting 1 (as $8 + 7 = 7 + 7 + 1$, or $8 + 7 = 8 + 8 - 1$) but others associate these combinations with their sums at sight.

Pupils rarely originate these indirect methods themselves. On the other hand, they can usually be traced to the early influence of a teacher, a parent, or an older brother or sister at home. Unfortunately, the methods which many primary teachers use are often directly responsible for the inefficient procedures which have been described. First and second grade teachers frequently follow the path of least resistance by deliberately instructing their charges to count objects and domino dots and to follow the other circuitous methods referred to in the preceding paragraph. Results are speedily obtained by these methods but, unfortunately, they produce habits which must later be broken up or they will encumber the pupil throughout his life. Far better no instruction at all in the first grade or in the second grade than instruction of this kind!

It behooves the primary teacher, then, to see to it that first impressions are correct impressions and that early practices lead to sound habits. The preceding chapter was written with this aim in mind. Let us now see how we can plan further work in addition so that the pupil's habits will be economical habits and his responses correct.

EASY COLUMN ADDITION

When pupils have learned the fundamental facts in addition, they are ready for examples in short column addition. One need not wait until the entire 45 addition combinations have been presented but may quite well introduce work in easy column addition after the 25 combinations whose sums do not exceed 10 have been taught. The teacher must, of course, be careful not to provide examples whose sums exceed 10. A very large number of

such examples may be arranged. One can, for instance, arrange 120 different examples in column addition with three digits in a column, the sums not exceeding 10. The following 31 combinations of three digits make the 120 examples when each combination is taken in all its possible arrangements.

1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8
—	—	—	—	—	—	—	—
1	1	1	1	1	1	1	1
2	2	2	2	2	2	3	3
2	3	4	5	6	7	3	4
—	—	—	—	—	—	—	—
1	1	1	1	2	2	2	2
3	3	4	4	2	2	2	2
5	6	4	5	2	3	4	5
—	—	—	—	—	—	—	—
2	2	2	2	2	3	3	
2	3	3	3	4	3	3	
6	3	4	5	4	3	4	
—	—	—	—	—	—	—	

Each combination like the first, where all three digits are the same, will, of course, make but one example. There are three of these in the list, making three examples. Each combination like the second will make three examples. They are:

1	1	2
1	2	1
2	1	1
—	—	—

Since there are 17 of these in the list, they will make a total of 51 examples. Each combination like the second in the second line will make six examples. They are:

1	1	2	2	3	3
2	3	1	3	1	2
3	2	3	1	2	1

Since there are 11 of these in the list, they will make, in all, 66 examples. We have, then, for the list of 31 combinations of three digits, $3 + 51 + 66 = 120$ examples.

The teacher may well give each of these 120 examples for practice. They should not, of course, be given in the systematic order in which they are arranged above but should be presented in a random order.

The remaining three-digit combinations. When the entire 45 addition combinations have been taught, three-digit column addition may be extended to include many more examples. There are 79 additional three-digit combinations which may be used, that is, 79 combinations of three digits in which the sum of the first two digits does not exceed 9. Of course, we cannot yet give three-digit examples in which the sum of the first two digits is 10 or more for the pupils have not yet learned higher decade addition. With the 79 three-digit combinations, a total of 204 arrangements are possible. In other words, when a pupil has learned the 45 addition combinations, he is ready to practise on 324 examples in column addition, three digits in a column, 120 of which have sums not exceeding 10 and in 204 of which the sums exceed 10. These 324 examples are made up from 110 three-digit combinations, in 31 of which the sums do not exceed 10. We give below the 79 three-digit combinations whose sums exceed 10.

1	1	1	1	1	1	1	1	1	1
1	2	2	3	3	3	4	4	4	4
9	8	9	7	8	9	6	7	8	9

1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	6	6	6	6	7
5	6	7	8	9	6	7	8	9	7

1	1	1	1	2	2	2	2	2	2
7	7	8	8	2	2	2	3	3	3
8	9	8	9	7	8	9	6	7	8
–	–	–	–	–	–	–	–	–	–
2	2	2	2	2	2	2	2	2	2
3	4	4	4	4	4	5	5	5	5
9	5	6	7	8	9	5	6	7	8
–	–	–	–	–	–	–	–	–	–
2	2	2	2	2	2	2	2	3	3
5	6	6	6	6	7	7	7	3	3
9	6	7	8	9	7	8	9	5	6
–	–	–	–	–	–	–	–	–	–
3	3	3	3	3	3	3	3	3	3
3	3	3	4	4	4	4	4	4	5
7	8	9	4	5	6	7	8	9	5
–	–	–	–	–	–	–	–	–	–
3	3	3	3	3	3	3	3	4	4
5	5	5	5	6	6	6	6	4	4
6	7	8	9	6	7	8	9	4	5
–	–	–	–	–	–	–	–	–	–
4	4	4	4	4	4	4	4	4	
4	4	4	4	5	5	5	5	5	
6	7	8	9	5	6	7	8	9	
–	–	–	–	–	–	–	–	–	

Again, we would remind the teacher that these combinations should be presented in random order and not in the order in which they are listed.

Adding up or down. As soon as the pupils begin column addition with examples whose sums exceed 10, the matter of adding up or down becomes significant. It will be observed that the three-digit combinations listed above have been arranged under the assumption that the pupil will add down. If the teacher desires that the pupils shall add up, most of the exam-

ples will have to be rearranged. In the first example, for instance, if the pupil adds up, he will think "10, 11," but $10 + 1$ is a combination which he has not had. If he were to add up, the combination would have to be arranged like this,

$$\begin{array}{r}9\\1\\\underline{1}\end{array}$$

There is little scientific evidence as to whether it is better for a pupil to add upward or downward. Buckingham,[1] in a recent discussion of the subject, points out that what is most needed is "An experimental attack upon the question to see whether under controlled conditions children learn more quickly and more satisfactorily by one method than by the other." He states, however, that "downward adders appear rather more frequently than the upward adders to have arrived at their preference through independent reason." The principal reasons were as follows:

1. "In downward addition the eye moves in the direction of ordinary reading."
2. "We write figures downward."
3. Adding downward, we arrive "at the point where the figure is to be written precisely at the moment when the sum is secured."
4. In carrying "upward addition introduces a rather complex series of eye-movements."

Further practice in column addition. Examples in which four or more digits are presented in a column may be easily arranged. In the case of a four-digit column, assuming that the pupil adds downward, the sum of the first three digits, counting down from the top, must not exceed 9. The fourth digit can be any one of the nine digits. It is probably not worth while for pupils to add columns containing more than three or four digits until training in higher decade addition has been provided.

Zeros in column addition. While the pupils are learning to do easy column addition, zeros may be introduced, *provided that there is some real setting for the examples.* A good plan is to let

[1] Buckingham, B. R. "Adding Up or Down: A Discussion", *Journal of Educational Research,* XII: 251-261, November, 1925.

the column represent scores on games, as the bean bag game. If the board on which the bean bags are thrown is so arranged that only scores represented by small one-digit numbers can be made, the scores may be set down in a column and added, a zero being recorded when the pupil makes no score. The pupil will simply be taught to skip zeros in adding a column of scores.

Problems and column addition. Easy problems may be formulated by both the teacher and the pupils, the solution of which requires the ability to add easy columns. Suggestions for formulating problems have been given in the preceding chapter.

HIGHER DECADE ADDITION

The need of specialized habits. We have seen that we must teach a pupil both $8 + 5$ and $5 + 8$. Experiments show that if a pupil knows the one form of this combination, we cannot assume that he will respond correctly to the other form also. In the same manner, we need to provide that the pupil's knowledge of the combinations be carried to higher decades—that he be taught $18 + 5$, $28 + 5$, etc. This higher decade addition must follow the teaching of the primary addition facts and must precede column addition, except in columns where the sum of all digits but one is not greater than 9. For example, a pupil who had not been taught higher decade addition could add the first example on the left without difficulty but would be unable to do the second, assuming that he adds downward in each case.

4	5
1	7
2	4
8	9
–	–

The number of higher decade combinations. If we combine the digits 1 2 3 4 5 6 7 8 9 with 10, 11, 12, etc., to 90, and with numbers in the 90's so that the sum does not exceed 99, we shall have a total of 765 combinations. There will be nine combinations for each number from 10 to 90 inclusive, making a total of 81×9, or, 729 combinations. With 91, we shall combine each digit from 1 to 8, inclusive, adding eight combinations; combination of the digits 1 to 7 with 92 adds seven more; and

so on until the last combination, 98 + 1, is reached. There will be a total, then, of $729 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 765$ combinations.

But these 765 combinations are not equally important. In column addition, except in a few specialized vocations, we rarely get sums in the 90's, the 80's, the 70's, or even in the 60's, the 50's, or the 40's. Most of these higher decade combinations can be safely omitted from the list upon which we must give special practice, particularly in the primary grades.

Osburn [2] recommends that we teach those higher decade additions whose sums do not exceed 39. Of these, there are 225—90 in the teens, 90 in the twenties, and 45 in the thirties. We may conveniently divide these 225 combinations into two groups: (1) those whose sums are in the same decade as the larger addend; and (2) those whose sums are in the decade next higher than that of the larger addend. There will be 135 of the former and 90 of the latter. The skill called for in the latter group is often referred to as "bridging the tens," because one uses his knowledge of a fundamental combination in bridging over from one decade to the next. As a convenience in organizing our lessons we may, then, divide the 225 combinations into groups, as follows:

Groups	Combinations in the			Total
	Teens	Twenties	Thirties	
Not bridging tens.....	45	45	45	135
Bridging tens..........	45	45	..	90
Total	90	90	45	225

The responsibility of teaching these 225 higher decade combinations cannot be treated lightly. If we would provide for efficient addition habits in our pupils, we must see to it that

[2] Osburn, Worth J. *Corrective Arithmetic.* Houghton Mifflin Company, 1924, p. 17.

their fundamental skills are extended to include additions in higher decades. Many pupils who have not developed counting habits in connection with the primary addition facts fall readily into these habits in adding columns. In testing his own students —prospective teachers—the writer has frequently discovered individuals who showed no evidence of counting in adding examples like the first one on the left, but who counted in each step after the first in adding examples like the second on the left—examples in which there were three or more addends. Many people in doing two-addend examples will write the sums without hesitation; but in adding examples containing more than two addends, they will pause after the first combination and count to get the sum made by the next digit. Thus, in the second example, above, they will think "14" readily enough and then count from 14 to 23, from 23 to 25, and finally from 25 to 32. Such individuals often leave their tracks behind them in the form of pencil dots sprinkled along the column, indicating that the counting was accompanied by tapping the pencil on the paper. The counting habit here, as in the primary facts, sometimes mars the appearance of one's paper, always consumes more time, and usually is a fruitful source of errors.

467	26
395	48
—	39
	82
	77
	—

Naturally, pupils will occasionally encounter columns whose sums are greater than 39, meaning that higher decade combinations, other than the 225 referred to, will sometimes be needed. After the 225 have been well taught and frequently practised in examples, a miscellaneous few running into the forties and the fifties may be introduced and the resemblance to those already learned pointed out. Since this resemblance is close, we may hope for considerable transfer here, particularly if the 225 have been presented as an extension of the fundamental facts, in the manner suggested in later pages.

Addition in multiplication. There are, however, a few higher decade combinations whose sums are greater than 39 which

should be given special attention. These are those which are needed in carrying in multiplication. In the example on the right, for instance, the pupil thinks, in connection with the last step in multiplication, "eight sixes are forty-eight and three make fifty-one." Here he needs the higher decade combination 48 + 3.

$$\begin{array}{r} 647 \\ 8 \\ \hline 5176 \end{array}$$

There are 36 numbers, in addition to zero, which appear as products of pairs of one-digit numbers in multiplication. They are: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, 32, 35, 36, 40, 42, 45, 48, 49, 54, 56, 63, 64, 72, 81. Twenty-five of these are less than 40; eleven are equal to or greater than 40. We are concerned here with the eleven which are greater than 39 and with 35 and 36 to the extent to which the carry numbers which may be added to them make 40 or more.

The greatest carry number which may be added to a product is always one less than the larger of the two factors making the product. Thus, the largest carry number which may be added to 35 is one less than 7 (the larger of the two factors which make 35), or 6. This statement is offered without proof but the reader may empirically satisfy himself that the statement is true by multiplying 199, 299, 399 . . . 999 by each number from 2 to 9, inclusive.

The higher decade additions, in addition to the 225 already provided for, which must be taught to prepare the pupil for carrying in multiplication, may be stated as shown in Table III. To the product, 56, for example, we may have to add any carry number from 1 to 7, inclusive, meaning seven higher decade combinations to be taught. It will be seen that a total of 87 higher decade combinations is added to our list of 225 if we are to make proper provision for carrying in multiplication.

In the last column of the table we have given the number of higher decade additions in each group which involve bridging tens. It will be seen that 35 of the 87 are in this class. It will be recalled that 90 of the 225 higher decade combinations previously discussed involved bridging tens. So we have

TABLE III. *Higher Decade Addition Combinations Needed in Carrying in Multiplication*

Product	Carry Numbers	Number of Combinations	Bridging Tens
35	5-6	2	2
36	4-8	5	5
40	1-7	7	0
42	1-6	6	0
45	1-8	8	4
48	1-7	7	6
49	1-6	6	6
54	1-8	8	3
56	1-7	7	4
63	1-8	8	2
64	1-7	7	2
72	1-8	8	1
81	1-8	8	0
Total		87	35

a total of 312 higher decade combinations to be taught; in 125 of these, the pupils must bridge tens and in 187 the sum is in the same decade as the larger addend.

How to teach higher decade addition. In teaching higher decade addition, the important point to be kept in mind is that each higher decade combination should be presented as an extension of the corresponding fundamental addition fact already learned. In other words, the higher decade combinations should not be treated as so many individual, isolated addition combinations but should be closely associated with the primary addition facts which the children have mastered.

Teach first the 135 higher decade addition combinations whose sums do not exceed 39 and which do not involve bridging. Then follow with the 90 whose sums are less than 40 and in which the pupil must learn to bridge. After these have been learned, the 87 additional combinations used in multiplication may be presented—first the 52 which do not involve bridging and finally the 35 which include bridging.

The latter group of 87 higher decade combinations should be presented considerably later than the other 225. They will not be needed until the class is ready to take up multiplication with carrying. They should, then, be presented *after* the multiplication combinations have been learned and *before* multiplication with carrying is introduced.

Let the first approach to higher decade addition be through a real situation which the pupils have experienced or through a problem making an imaginative appeal. We may, for example, start with 22 + 6. The classroom presentation may proceed as follows:

Teacher: "Harry sold 22 papers before supper and 6 more after supper. How many did he sell that evening? Here is a problem in which we must add 22 and 6. To add 22 and 6, think first of 2 and 6. How many are 2 and 6, William?"
William: "Eight."
Teacher: "If 2 and 6 are 8, how many are 22 and 6? Eleanor." (Eleanor is a bright girl who the teacher believes will see the connection between "2 and 6" and "22 and 6" readily.)
Eleanor: "Twenty-eight."
Teacher: "Yes. Then to add 22 and 6, we think first, '2 and 6 are 8' and then, '22 and 6 are 28.' How many are 32 and 6, Alice?"
Alice: "Thirty-eight."
Teacher: "How many are 12 and 6, John?"
John: "Eighteen."
Teacher: "Remember, then, to add 22 and 6, or 32 and 6, or 12 and 6, we first think of 2 and 6. But 2 and 6 are 8, so 22 and 6 are 28, 32 and 6 are 38, and 12 and 6 are 18."

Other problems involving these three higher decade additions are then introduced and the three combinations are called for repeatedly, until each pupil in the group has had at least one opportunity to respond. This work should proceed very slowly at first but may go forward more rapidly after the first few lessons.

Drills in higher decade addition. Drills in higher decade addition may be both oral and written. The oral drills may be conducted with flash cards, as were the drills in the fundamental combinations, or the teacher may simply dictate the

combinations, calling upon individual pupils for responses. In either case, the teacher should call the pupil by name *before* giving the combination so that the pupil may give his response in the shortest time possible.

Written drills are better than oral drills in that they make possible a larger measure of participation on the part of each pupil. For written drills, it is best to prepare mimeographed sheets of higher decade combinations, seeing to it that the practice is distributed fairly evenly over the combinations taught but that the more recently presented combinations are emphasized and that those which the pupils have found particularly difficult are given extra practice. If duplicating facilities are not available, the combinations may be written on the blackboard and copied by the pupils, or written by the pupils from dictation. Whatever plan is followed, the pupils should all begin adding at the same time and a time limit should be imposed in order that counting may be discouraged. If the fundamental combinations have been well taught and the higher decade combinations have been presented as an extension of these fundamental combinations, there should be little difficulty in avoiding the counting habit and in training pupils to give immediate responses.

The higher decade combinations should be written in the vertical form with the smaller number underneath, like this, $\begin{array}{r}32\\6\\\hline\end{array}$ $\begin{array}{r}24\\9\\\hline\end{array}$. This is the order in which the numbers will come to the pupils in column addition and in carrying in multiplication, if they add downward.

Further work in column addition. It is well that the primary teacher keep in mind the fact that higher decade addition is taught primarily as a means of bridging the gap between the addition combinations and column addition. There are, to be sure, occasional real problems requiring higher decade addition, such as that which we used to introduce the teaching of the

subject, but the prime use of higher decade addition lies in column addition.

As higher decade addition combinations are taught, then, they should be used in column addition examples as soon as possible. When we have taught 11 + 5, for instance, the column addition example on the right may be given (assuming that the pupil adds downward) as a means of putting to use his recently acquired skill.

6
2
3
5
—

Further work in column addition should be very carefully planned so as to give practice on each of the higher decade combinations which have been taught. Furthermore, these column addition examples should be so constructed as to give practice on each of the fundamental addition combinations. The mere random arrangement of digits in columns may mean that some of these combinations, primary and higher decade, are given an undue proportion of attention in the drill lessons.

This means that a series of practice sheets should be arranged, providing drill in the fundamental combinations and in the higher decade combinations in systematic manner. The following sets of column addition examples have been devised for this purpose. The first set contains 45 examples and provides practice on each of the 81 primary facts and on each of the 45 higher decade combinations whose sums are in the teens.[3] Such practice is provided by 36 column addition examples of four digits each and 9 of three digits each. Zeros have been omitted for the reasons stated in the preceding chapter. The assumption is again made that the pupil will add downward.

The second set also contains 45 examples and is designed to give practice on those 45 higher decade combinations whose sums are in the twenties and which involve bridging. This set includes a review of those 45 primary facts whose sums are 10 or more and of the 45 higher decade combinations whose sums

[3] It is not intended that all 45 examples in this set or later sets be given for a single drill lesson. The sets of practice examples in this book are intended for the teacher's use in such numbers and at such times as she thinks best. Later, an entire set may be used as a review test.

PRACTICE EXAMPLES IN COLUMN ADDITION. SET I

Higher Decade Combinations in the Teens

3	2	1	1	1	7	5	5	3
5	5	7	1	2	1	2	3	6
6	5	2	8	7	3	6	7	5
3	4	7	2	1	6	3	2	4
—	—	—	—	—	—	—	—	—
2	1	2	1	8	6	6	4	2
6	8	1	3	1	2	3	4	7
4	1	8	6	2	5	6	8	3
5	8	1	3	7	4	3	1	6
—	—	—	—	—	—	—	—	—
3	1	7	4	5	2	4	1	2
1	4	2	5	4	2	1	5	3
7	5	4	7	8	8	6	4	7
2	4	5	2	1	1	3	5	2
—	—	—	—	—	—	—	—	—
3	3	5	1	2	4	3	4	6
3	2	1	6	4	3	4	2	1
8	8	5	3	6	7	8	7	4
1	1	4	6	3	2	1	2	5
—	—	—	—	—	—	—	—	—
1	3	5	4	2	6	9	7	8
9	9	9	9	9	9	9	9	9
9	7	5	6	8	4	1	3	2
—	—	—	—	—	—	—	—	—

are in the teens. In other words, a pupil makes three additions in each of these 45 examples. The first of these additions is a primary fact with a sum equal to or greater than 10; the second is a higher decade combination with a sum in the teens; the third is a higher decade combination with a sum in the twenties. No primary fact or higher decade combination occurs twice in this set. If the purpose of the set is to be realized, the pupil must add downward.

The third set contains 45 column addition examples providing practice on each of those higher decade combinations whose

PRACTICE EXAMPLES IN COLUMN ADDITION. SET II

Sums in the Twenties

9	2	7	9	6	5	7	8	6
2	8	8	4	6	5	4	4	8
5	5	1	3	1	2	6	5	1
8	9	4	6	9	8	5	7	5
—	—	—	—	—	—	—	—	—
7	8	5	7	6	3	6	5	9
3	2	6	9	7	7	5	7	8
1	7	2	1	6	9	7	7	2
9	4	7	8	7	4	6	1	9
—	—	—	—	—	—	—	—	—
7	4	4	3	6	4	8	3	8
7	6	9	9	9	7	5	8	6
3	8	1	6	3	1	2	8	4
6	7	8	2	4	9	7	6	3
—	—	—	—	—	—	—	—	—
8	6	4	9	5	9	9	9	1
8	4	8	6	8	5	9	3	9
3	4	2	2	4	2	1	4	3
8	7	6	9	3	7	2	9	8
—	—	—	—	—	—	—	—	—
8	9	9	8	8	7	2	5	7
3	7	1	7	9	6	9	9	5
3	2	6	4	1	5	4	5	3
9	9	5	3	8	5	6	5	8
—	—	—	—	—	—	—	—	—

sums are in the twenties and which do not involve bridging. The exercise is so arranged that each of the 45 higher decade combinations specifically practised in the second set—those whose sums are in the twenties and in which the pupil must bridge—are reviewed here. A review is also provided for 12 of the higher decade combinations whose sums are in the teens and for 29 of the more difficult primary facts.

The fourth set is designed to practise those 45 combinations

PRACTICE EXAMPLES IN COLUMN ADDITION. SET III

Sums in the Twenties

6	9	7	6	9	7	8	6	9
8	7	5	6	4	6	3	7	9
6	5	8	9	9	7	9	8	2
3	6	1	2	2	4	2	1	7
—	—	—	—	—	—	—	—	—
6	7	8	8	9	5	9	8	7
9	9	8	9	5	9	8	9	9
8	9	4	6	9	7	5	8	6
1	2	5	3	2	4	6	1	3
—	—	—	—	—	—	—	—	—
7	9	7	9	8	8	9	9	8
8	9	8	9	6	8	7	6	9
9	6	5	3	8	7	8	7	4
2	3	6	7	1	4	1	4	5
—	—	—	—	—	—	—	—	—
					5	8	2	4
9	8	9	9	9	8	4	9	6
8	7	9	8	9	4	7	8	9
3	6	4	7	5	9	1	2	3
8	3	5	4	5	2	9	8	7
—	—	—	—	—	—	—	—	—
9	8	8	4	9	4	4	7	9
9	7	5	9	8	7	8	7	7
1	4	5	6	2	7	6	5	3
9	6	7	4	8	9	8	5	7
1	3	4	6	1	2	1	5	3
—	—	—	—	—	—	—	—	—

whose sums are in the thirties and in which the sum of all digits but the last is a number in the twenties. Bridging is, of course, required. The set is so arranged that drill is provided for forty of the primary facts (including all of the 36 whose sums are greater than 10); for 10 of the higher decade combinations whose sums are in the teens; for 30 of those whose sums are in the twenties and in which bridging is

necessary; and for 17 of the combinations specifically practised in the Set III (those in which the sums are in the twenties and in which no bridging is required). In the example on the right, for instance, the pupil is given practice on 4 + 7, on 11 + 8, on 19 + 9, and on 28 + 3. The last is, of course, one of the 45 combinations for which the set is particularly prepared.

4
7
8
9
3
—

PRACTICE EXAMPLES IN COLUMN ADDITION. SET IV

Higher Decade Combinations in the Thirties

5	8	9	8	6	9	7	9	6
8	8	9	9	7	8	8	9	8
8	9	6	9	9	9	8	7	9
9	5	9	6	9	4	9	7	8
—	—	—	—	—	—	—	—	—
7	8	9	9	9	9	5	8	9
9	7	8	5	9	7	9	9	7
9	9	8	9	8	8	8	7	9
8	6	6	7	5	7	8	8	9
—	—	—	—	—	—	—	—	—
3	6	5	9	8	4	8	6	7
9	9	6	6	5	8	3	5	4
7	8	9	7	6	9	9	8	9
9	4	6	5	8	7	8	7	8
9	8	7	9	4	5	8	8	4
—	—	—	—	—	—	—	—	—
8	4	6	8	7	7	5	4	7
6	7	6	4	5	7	7	9	6
8	8	9	7	8	6	9	9	7
5	9	6	9	7	8	5	6	7
7	3	5	6	3	2	9	7	6
—	—	—	—	—	—	—	—	—
9	6	3	3	2	7	9	5	9
2	4	8	7	9	3	4	5	3
8	9	5	7	3	9	3	8	3
6	5	7	9	7	9	6	9	5
4	5	6	3	8	1	7	2	9
5	9	1	8	3	7	2	6	4
—	—	—	—	—	—	—	—	—

PRACTICE EXAMPLES IN COLUMN ADDITION. SET V

Higher Decade Combinations in the Thirties

1	9	9	5	6	7	5	6	8
9	1	2	5	5	4	8	7	6
2	3	3	4	6	6	6	8	7
9	9	8	9	7	8	8	7	6
9	8	9	8	8	7	6	5	8
6	3	7	3	7	2	4	2	1

9	8	9	5	8	2	7	7	6
6	7	5	7	3	8	9	6	6
7	5	8	5	5	4	7	7	7
6	9	7	9	8	9	4	5	8
7	6	4	6	7	7	9	9	5
4	2	5	6	1	2	2	3	3

3	8	9	5	4	8	3	8	6
8	2	7	9	9	4	7	8	8
5	5	6	7	6	6	6	5	4
9	9	6	5	9	8	9	8	9
6	6	8	8	4	5	5	7	7
5	9	1	1	5	2	7	3	4

8	3	7	9	7	5	4	4	8
5	9	3	9	7	6	8	6	9
8	6	7	5	5	7	8	8	6
8	9	9	5	9	6	8	9	6
3	4	4	9	6	9	3	3	8
1	6	1	1	2	3	8	4	2

6	7	9	6	9	7	9	4	2
9	5	3	4	8	8	4	7	9
8	6	8	9	5	4	5	4	9
6	7	9	9	7	7	8	8	9
5	8	2	2	9	9	7	9	1
5	1	4	8	1	3	6	4	5

EXTRA ADDITION COMBINATIONS NEEDED IN MULTIPLICATION. SET I

No Bridging Required

54	64	40	81	63	42	54	42	63
1	5	1	2	2	3	2	4	3
40	81	72	63	40	72	42	81	54
2	3	1	4	3	2	5	4	3
54	72	40	81	42	63	45	63	40
4	3	4	5	6	5	1	6	5
72	54	81	56	40	64	81	45	72
4	5	6	1	6	1	7	2	5
56	40	72	45	81	64	56	42	45
2	7	6	3	8	2	3	1	4
64	72	64	48	42	63	81		
3	7	4	1	2	1	1		

EXTRA ADDITION COMBINATIONS NEEDED IN MULTIPLICATION. SET II

Bridging Required

45	36	49	64	48	35	63	48	45
8	4	2	7	2	5	8	6	5
63	48	36	48	56	56	54	49	54
7	5	6	3	6	4	7	4	6
35	54	56	36	48	49	45	49	56
6	8	5	7	4	5	7	1	7
49	36	64	48	36	45	72	49	
3	5	6	7	8	6	8	6	

and before multiplication with carrying is introduced. The exercise follows.

6 9 7 4 8 5

Multiply each of these numbers by 7 and add 1 to the product; add 2 to the product; add 3 to the product; add 4 to the product; add 5 to the product; add 6 to the product.

Multiply each of these numbers by 8 and add 1, 2, 3, 4, 5, 6, and 7 to the product.

Multiply each of these numbers by 9 and add 1, 2, 3, 4, 5, 6, 7, and 8 to the product.

This exercise provides for a total of 107 higher decade additions. Eighty-seven of these have sums greater than 40; the remaining 20 are reviews. Since 7×8 and 8×7, 7×9 and 9×7, and 8×9 and 9×8 occur, there are 19 higher decade additions in the exercise which are duplicated.

We have now made careful provision for practice on those higher decade addition combinations whose sums are less than 40 and those needed in carrying in multiplication. It is not to be inferred that we would limit the pupil's training to these. Other higher decade additions should be occasionally introduced, particularly in the oral drills, and the resemblance to those already learned pointed out if the pupil hesitates in his response. In the first three grades, however, there will not be frequent need for combinations whose sums are in the forties and higher decades. Further training of this type will be necessary in the intermediate grades (grades four, five, and six).

CARRYING

Addition ability a complex. It must now be apparent that addition is a complex of several simpler abilities or kinds of skills. Addition involves several specific types of response. We can never be sure that a pupil will meet a situation successfully unless he has been trained in each of the specific elements of response which the situation requires. So far, we have made

provision for teaching the fundamental addition facts and for extending the skills thereby acquired into the higher decades. The pupil must now be prepared for examples containing two or more columns in which the sum of one column exceeds nine.

The meaning of carrying. Other things being equal, new processes should be rationalized for the pupils; they should see reasons for the things which they are taught to do. Some teachers simply *tell* their pupils that when the sum of a column is greater than nine, the right-hand figure only should be written down and the other figure "carried" to the next column. Telling is likely to mean mere rote learning, however. Our objective should be the acquisition of skill *with understanding.*

A favorite device for teaching the meaning of carrying is the use of bundles of splints. Suppose that we begin with an example like $\begin{array}{r}18\\ \underline{14}\end{array}$. The number 18 is represented by a bundle of ten splints and eight loose splints, a device previously used in teaching number meanings. Thus, the fact that 18 is one ten and eight ones is impressed anew. Likewise 14 is represented by a bundle of ten splints and four splints more. When the two numbers are added we have the process concretely represented by two bundles of ten splints each and twelve splints more. But twelve splints will make a bundle of ten splints with two splints left over. The teacher proceeds to tie up ten splints into a bundle. We have in all, then, three bundles of ten splints each and two splints more, or 32 splints. Then 18 and 14 are 32.

After this concrete presentation, the situation may be reviewed as follows:

Teacher: "How many tens in 18?"
Pupil: "One."
Teacher: "How many units in 18?"
Pupil: "Eight."
Teacher: "How many tens in 14?"
Pupil: "One."
Teacher: "And how many units?"
Pupil: "Four."

Teacher: "If we add the eight units and the four units, how many shall we have?"
Pupil: "Twelve."
Teacher: "But how many tens in 12?"
Pupil: "One."
Teacher: "And how many units?"
Pupil: "Two."
Teacher: "Let us, then, put our one ten in with the other tens, making three tens, and write our two units under the 8 and 4. Then how many tens do we have?"
Pupil: "Three."
Teacher: "And our answer is 32, which is what we got with the splints. Let us remember, then, that if the sum of a column is 10 or more, we should write down the right-hand figure only and carry the other figure to the next column, if there is one."

The pupils then proceed to practise on other similar examples.

Another interesting method, similar to the use of splints, makes use of a problem involving the addition of sums of money. The lesson may proceed as follows: Mary had 18 cents and made 14 cents selling Christmas cards. How many cents had she then?

18 cents. This is the same as 1 dime and 8 cents.
14 cents. This is the same as 1 dime and 4 cents.
32 cents.

First we add 8 cents and 4 cents, getting 12 cents. But 12 cents is the same as 1 dime and 2 cents. So we write the 2 cents in the cents column and carry the one dime to the dimes column, making 3 dimes and 2 cents, or 32 cents. Coins may be used to objectively illustrate the process.[5]

In another scheme, the pupil is taught to break up the lower number of the two to be added, calling 14, for example, $10 + 4$. The 10 and 4 are then added separately as shown. The first step is, of course, easy and the second step is easily accomplished by virtue of the work in higher decade addition recently completed. The pupil is then told the usual carrying procedure and has confidence in it because it gives the same answer as that obtained by the other process. This method should be used

$$\begin{array}{r} 18 \\ 10 \\ \hline 28 \\ 4 \\ \hline 32 \end{array}$$

[5] The use of coins is probably to be preferred to the use of splints. Coins are more realistic and probably are less likely to suggest counting.

cautiously, if at all, because of the danger that pupils will use it as a crutch and delay learning the shorter and more useful procedure.

The first day that carrying is taught, the work may well be confined to carrying one, but pupils should quite early be given practice in carrying two and three. If all of their early work in carrying is confined to carrying one, they are likely to associate "carry" with "carry one" regardless of the example and will, from the force of habit, carry one in examples like that on the left.

24
16
17
38
—

Writing the number carried. Opinion differs as to the wisdom of writing the number which is to be carried. In examples such as will be taught in the primary grades—examples of few addends and few digits in each addend—writing the carry number will, perhaps, not be advisable. In the addition of long examples, however, it is probably best to write the carry number at the top of the next column. This not only permits one to take up his work where left off, if interrupted, but also makes it possible to check the columns separately to detect errors. The carry number will, of course, be added in at once in the next column. Whether examples long enough to justify writing the carry number should be given in the elementary school is again a question. Perhaps this crutch should not be used at all in the elementary school until its benefits have been experimentally determined.

Variety in addition practice. A large variety of kinds of addition examples may be prepared by varying the number of addends and the number of digits in each addend. If, for example, we have two, three, or four digits in an addend and two, three, four, five, or six addends, we can have fifteen different kinds of addition examples. Further variety may be introduced by using addends of different lengths in the same example.

It is usually necessary to prepare sets of addition examples, supplementing those provided by the textbook. Lack of space prohibits our supplying them here but the method of prepar-

ing them is similar to that employed in preparing the sets of examples in column addition. The teacher should keep in mind the following points:

1. Use a variety of examples, as suggested above.
2. Provide that the pupils shall frequently carry other numbers than one.
3. See that each of the 81 primary addition facts is given occasional practice.
4. Provide for occasional practice on each of the higher decade addition combinations which have been taught.
5. Use zeros frequently.

Zeros in addition. It will be recalled that in the plan for teaching the primary addition facts, in the preceding chapter, no provision was made for teaching the zero combinations. Zeros should be used in situations in which they normally occur. When pupils have learned the fundamental combinations and the higher decade combinations, however, and are ready for practice on addition examples in which there are two or more columns, they should be taught how to handle zeros in addition and should be given considerable practice with them. For a time, zeros should occur more frequently than any other digit in addition examples, not only to give the pupil needed practice in adding zeros but also because in certain types of addition work, as in adding amounts of money, zeros may actually occur more frequently than any other digit.

The meaning of zero must, of course, be clear. Zero means *not any* or *no*. Thus, zero apples means *no* apples or *not any* apples. In adding, then, pupils are taught to simply disregard zeros in the columns of digits. With an explanation of the meaning of zero and plenty of practice in the addition of zeros, pupils should not have difficulty with them.

Addends of unequal length. Throughout the grades, pupils should have occasional practice in the addition of addends of unequal length. Tests show that pupils become confused and

make mistakes when compelled to jump gaps in addition columns unless they have received training in this type of work.

```
4625
  14
 372
   6
  14
----
```

In the example on the left, after the first column has been added, 1 has been written below the line, 2 has been carried to the next column and this column has been added down to and including the 7, the pupil will probably pause, wondering what to do with the blank space. While hesitating he may fail to retain in mind the partial sum, 12, and hence may be compelled to repeat his work to this point or he may make an error. Some pupils, rather than jump the gap from the 7 to the 1, will add in the 6 of the first column.

When examples of this kind are first presented, teachers sometimes fill in the vacant places with zeros, making the example appear as shown.

```
4625
0014
0372
0006
0014
----
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This device may be quite harmless as a means of showing that certain orders are unoccupied but it should be used sparingly, if at all, lest the practice become a permanent one with some pupils. Like most crutches, it may quite well do more harm than good.

Teaching children to add addends of unequal length is like teaching them to add zeros. In adding the second column of the example above, the pupils are shown that there are no tens in the fourth number and that we simply jump to the 1 as though the space next to the 6 were occupied by a zero. It is assumed, of course, that the addition of zeros has preceded the addition of addends of unequal length.

The importance of writing figures that are neat, of uniform size, and in straight columns will be apparent here. Careless

figure writing is often responsible for the mistakes of children who have no trouble with printed examples composed of addends of varying lengths. The teacher may discover, however, that pupils who persistently make mistakes in examples of this type are suffering from visual defects as a result of which they are unable to follow with the eye columns of digits containing blank spaces. Any pupil who continues to have trouble with this class of examples should be examined by an expert oculist.

Addition objectives in the primary grades. In a later chapter we shall consider the matter of objectives in some detail while discussing standards of achievement. It may be reiterated at this point, however, that the primary teacher should strive to develop, in the pupil, habits of response which are economical of time and energy and which yield accurate results. If the bonds pertaining to the primary facts, the higher decade combinations, carrying, zeros, and blank spaces in columns, are well fixed and ready to act, pupils should add examples like that on the right rapidly and accurately.

$$\begin{array}{r} 340 \\ 597 \\ 609 \\ 45 \\ 283 \\ 7 \\ \hline \end{array}$$

From the first, the teacher should strive to develop language habits in addition which will not prevent the development of proper speed later. In adding the first column of the example above it is wasteful to get in the habit of saying, "seven and nine are sixteen, sixteen and five are twenty-one, twenty-one and three are twenty-four," etc. It is better from the first to say simply, "sixteen, twenty-one, twenty-four, thirty-one." After the pupil's responses become implicit, that is, unaccompanied by discernible movements of speech organs, they will persist in much the same way that they were made when overt. Then, as the combinations become better and better learned, the pupil's speed will be limited by these elaborate implicit responses.

Speed versus accuracy. The reader should not conclude from the preceding paragraph that speed is comparable in importance to accuracy. Accuracy should be the primary objective. It is important to remember, however, that the same habits

which yield accurate results will also produce speed, and *vice versa*. Elaborate and roundabout methods reduce speed and they also produce errors. If the pupil is well taught, speed will probably take care of itself and will be limited largely by the limitations of the individual's reaction time. Lack of speed is not in itself necessarily a fault, but it may be indicative of serious faults. As a means of discouraging counting and as an aid in breaking up counting habits, the pupil should be urged to form the economical habits which have been suggested and to add as rapidly as is consistent with *his* progress.

Checking results. The habit of checking results in all computation is of fundamental importance. The place to develop and fix this habit is in the primary grades. In the intermediate grades, pupils are often supplied with textbooks containing answers and, as a result, there is little incentive for checking. Whatever we may say of the use of answers in the intermediate grades, we can most heartily condemn them in the primary grades where the curriculum is largely made up of simple problems and examples in the fundamental operations with integers. Because the habit of checking should be established early and because there is an incentive for checking where pupils are not supplied with answers, it is incumbent upon the primary teacher to see to it that her pupils regularly and systematically check all their addition examples, except in tests where insufficient time is allowed.

Checks in the primary grades should be very simple and easily applied. In addition, there are probably only two checks which the primary teacher will wish to use. They are (1) repeating the addition in the same direction, ordinarily downward; (2) repeating the addition in the opposite direction, ordinarily upward.

Repeating the addition in the same direction is the more easily applied check, for it is simply a repetition of what the pupil has already done. Its objection lies in the fact that when one has made an error, he tends to repeat that error on

the second and later addings. Since the sum, up to the point where the error was made, is the same each time, adding the next digit is likely to give the result secured before on account of the association which has been established between that result and the previous sum.

It is better, then, to check by adding in the opposite direction. If the recommendations of the previous pages have been followed, the pupil's first adding has been downward. He will now check by adding upward. This combining of the digits in a different order gives different partial sums than those secured by the first adding and thus makes it improbable that a former error will be repeated.

In examples like the one on the right, however, this check cannot be applied until higher decade addition has been taught. In such cases the only check which can be used satisfactorily is a repetition of the former adding process. It is quite important that the teacher should not lose sight of details such as this in planning practice exercises in the fundamental operations.

$$\begin{array}{r} 2 \\ 6 \\ 7 \\ \hline \end{array}$$

EXERCISES

1. Why should pupils not be taught to add 9 by adding 10 and subtracting 1?

2. Give other examples of roundabout methods which you have seen pupils or adults follow in addition.

3. Thorndike says, "Teachers are tempted to be short-sighted, sacrificing the future for the present." Show how this statement may apply to the teaching of addition.

4. Do you add up or down? Is the other method awkward for you? What arguments, other than those stated, can you give for adding downward? What arguments are there for adding upward?

5. Under what circumstances should zeros be introduced into single column addition?

6. Formulate four problems which you would use for single column addition?

7. If a pupils knows 7 + 6, why is it necessary to teach him 27 + 6, 17 + 6, etc.?

8. What is meant by the expression, "bridging the tens"?

9. If we teach just the 225 higher decade combinations listed, how can we be sure that the pupil will know that 78 and 4 are 82 if he meets this combination a year or more later?

10. By trying a number of examples, persuade yourself that the number carried in multiplication is never as large as the multiplier.

11. Why should practice in higher decade addition ordinarily be given in examples in column addition?

12. What changes should be made in the five sets of practice examples in column addition if the pupil is to add upward?

13. Can you give further suggestions for teaching the meaning of carrying?

14. How many kinds of addition examples can you make, having not over four digits in an addend and not over six addends in an example?

15. Why is speed important in addition?

16. Discuss the importance of checking computations.

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CHAPTER VI

THE TEACHING OF SUBTRACTION

Subtraction does not suffer so seriously from time-consuming, indirect methods as does addition. Pupils do, indeed, form poor habits in subtraction and, in some cases, invent strange ways for finding differences or remainders, but practices of this kind are neither so numerous nor of such frequent occurrence as are similar practices in addition.

The problems pertaining to the teaching of subtraction are fewer and simpler than those incident to the teaching of addition, because, in a subtraction example, we have but two quantities to deal with—the minuend and the subtrahend. On the other hand, the number of addends in an addition example is usually greater than two and may run to any number, although examples containing more than six addends are rarely assigned in the primary grades.

Subtraction examples without borrowing. To give practice on the subtraction facts which have been learned, and to acquaint children with the general nature of subtraction examples, considerable practice should first be provided with examples in which each subtrahend digit is smaller than, or equal to, the minuend digit directly above. In such examples, zeros cannot be used in the minuend unless zeros occur in the corresponding positions in the subtrahend.

These examples may include the two types of zero difficulties which occur in subtraction examples without borrowing—that in which a number is subtracted from itself, leaving zero, and that in which zero is subtracted from a number leaving the number.

The examples should, of course, gradually increase in length

and difficulty. At first, we may have but two-digit minuends and subtrahends, then three digits in each, four digits in each, etc. Early practice on minuends and subtrahends of two and three digits should be confined to the 36 primary subtraction facts in which no zeros occur in minuend, subtrahend, or remainder. Then the zeros may be introduced—first, a number less itself leaving zero, then, a number less zero leaving the number, and, finally, both types of zero difficulty.

The teacher may easily arrange sets of practice exercises distributing the drill over these combinations. Many such sets should be prepared and used. The practice should be distributed rather evenly over the combinations included, giving, however, somewhat more practice to the more difficult than to the easier. A few such sets have been prepared and are reproduced here for illustrative purposes.

The first set contains 18 examples made up of two-digit minuends and two-digit subtrahends. Each of the 36 primary facts referred to is included once in this set.

PRACTICE EXAMPLES IN SUBTRACTION. SET I

Minuends Not Exceeding 9

78	93	69	58	49	86	82	69	59
46	71	43	25	35	71	21	51	46
47	77	58	97	46	68	73	59	98
13	62	31	85	22	34	12	14	23

The second set contains 18 examples in which the minuends and subtrahends have three digits each. In this set, each of the 36 primary facts is practised and the more difficult 18 are included twice.

The third set includes the same 36 primary facts practised in Sets I and II and the 19 zero facts also. Five of these 55 facts occur twice in this set. Specific instruction in the zero difficulties will, of course, precede the use of this set. In this

PRACTICE EXAMPLES IN SUBTRACTION. SET II

Three-Digit Numbers

849	798	596	687	683	885
526	335	471	422	272	362
799	989	795	976	897	489
347	662	651	563	442	123
747	947	689	856	387	792
534	214	571	132	135	181

instruction, the two principles which have been stated—that *any number less itself leaves zero,* and that *any number less zero leaves the number itself*—will be brought out through concrete presentation and applied to each of the zero facts concerned. The former principle, with the nine facts to which it applies,

9	8	7	6	5	4	3	2	1
—9	—8	—7	—6	—5	—4	—3	—2	—1

may be developed in one lesson while the second principle with its 10 facts,

9	8	7	6	5	4	3	2	1	0
—0	—0	—0	—0	—0	—0	—0	—0	—0	—0

may be presented in another lesson. Zeros in those subtraction examples which do not involve borrowing should not be difficult for children to handle, if these combinations are left out of the original list of addition and subtraction facts and are presented with careful explanations immediately preceding their use in examples.

These three sets of examples are not intended to afford sufficient practice on the types of subtraction which they include

PRACTICE EXAMPLES IN SUBTRACTION. SET III

Zero Facts Included

6958	7948	8793	2851	5967
4621	6903	2480	1631	4023
9483	4079	3986	7285	9746
3401	1021	2580	5045	4721
8617	9365	8256	7498	9718
5600	2331	7205	1375	2403

but are offered as samples of the kind of practice material which should be used. Each may be used more than once, to be sure, but it is better for the teacher to prepare other sets, similar to these, that a rich variety of practice material may be at hand. Four-digit minuends and subtrahends without the zero difficulties may constitute a set. The zero difficulties in Set III will need much more practice than is afforded by doing the examples of this set.

In the next stage of the pupil's progress, examples similar to those in the sets given will be used, but in the last subtraction in an example, a single-digit number will be subtracted from a two-digit number. This will give practice on the remaining 45 subtraction facts and will be a good introduction to subtraction examples involving borrowing. As indicative of the kind of subtraction examples which may be used to give practice on these 45 facts, Set IV has been prepared. In the 45 examples of Set IV each of the 45 facts referred to occurs once. It will be noted that in each of the examples in this set the minuend has four digits and the subtrahend, three. Thus, each example gives practice on two facts, in addition to that for which the example is specifically prepared. Each of the 55 subtraction facts whose minuend is not greater than 9 occurs once, but the 35 most difficult of these (according to Clapp) occur twice.

PRACTICE EXAMPLES IN SUBTRACTION. SET IV

No Borrowing Required

1389 562	1097 465	1426 904	1184 851	1659 720	1257 834	1513 703	1087 903	1256 352
1189 904	1078 638	1775 870	1124 521	1496 780	1097 192	1383 940	1162 212	1548 903
1218 517	1097 240	1849 947	1368 402	1083 812	1493 611	1179 306	1068 320	1516 806
1390 850	1258 927	1648 935	1274 620	1489 532	1197 756	1326 704	1089 763	1127 415
1457 806	1236 413	1769 917	1196 603	1295 731	1623 810	1074 512	1585 624	1376 645

METHODS OF SUBTRACTION

Before taking up the teaching of subtraction with borrowing, let us consider the much disputed question as to which is the best method of subtraction. Four different methods have been advocated and are taught in the schools. Each has its ardent supporters and its bitter opponents.

The four methods. The four methods are called by various names, among which are the following:

1. Take-away-borrowing.	46045
2. Take-away-carrying.	29478
3. Addition-borrowing.	———
4. Addition-carrying.	16567

The first method is very often referred to as the *method of decomposition* and the second method, the *method of equal additions*. In each case, one subtracts by taking-away but in the former case, to compensate for increasing the 5 to 15, the 4 is

reduced to 3, while in the latter case, the 7 is increased to 8. In the former case, the 4 in tens' place is broken up or "decomposed" and 1 of the tens is combined with the 5, leaving 3 tens. In the second case, to compensate for adding a ten to 5, an equal amount is added to the tens' place in the subtrahend and the 7 tens becomes 8 tens. In the former case, one thinks in the above example, "8 from 15 leaves 7, 7 from 13 leaves 6, 4 from 9 leaves 5, 9 from 15 leaves 6, 2 from 3 leaves 1." In the latter case, one thinks, "8 from 15 leaves 7, 8 from 14 leaves 6, 5 from 10 leaves 5, 10 from 16 leaves 6, 3 from 4 leaves 1." In examples in which there is no borrowing, these two methods are identical.

Both the third and the fourth methods are sometimes referred to as the Austrian method, although this term seems to be more frequently applied to the fourth method than to the third. These methods are also identical in examples in which there is no borrowing. In both, one *adds* to the subtrahend whatever number is necessary to get the minuend. The third method is like the first in that decomposition is employed, while the fourth method is like the second in that equal additions are made in the minuend and subtrahend. In the third method, one thinks, "8 and 7 are 15, 7 and 6 are 13, 4 and 5 are 9, 9 and 6 are 15, 2 and 1 are 3." In the fourth method, one thinks, "8 and 7 are 15, 8 and 6 are 14, 5 and 5 are 10, 10 and 6 are 16, 3 and 1 are 4."

Concerning these four methods, there are two principal disputes. The first has to do with the relative merits of decreasing the minuend and increasing the subtrahend to compensate for borrowing. This dispute evidently applies to methods 1 and 3 as opposed to methods 2 and 4. The second dispute has to do with the relative merits of teaching subtraction as a "take-away" process and as an addition process. In this dispute, methods 1 and 2 are allied against methods 3 and 4.

The first dispute. As to the first dispute, there seems to be no scientific evidence available which shows clearly the superi-

ority of one method over the other. Ballard,[1] as the result of a limited experiment, reported some superiority for the method of equal additions. McClelland [2] found that the method of equal additions appeared to be superior for quick learning of subtraction but that the method of decomposition resulted in greater speed after long practice. Winch [3] claimed some slight superiority for the method of equal additions. Although these English investigators produced results which seem to favor the method of equal additions, the evidence is by no means sufficient to convince us that this method should be uniformly adopted for American schools. Excellent arguments have been advanced on both sides of the question. Whichever side one chooses, he will find himself in good company.

The method of decomposition seems to have a little advantage in that it helps to teach the pupil our system of decimal notation. When the minuend digit is smaller than the subtrahend digit in units' place, he "borrows" one ten, changes it to ten ones, combines the ten ones with the ones already in the minuend and proceeds to subtract. This method seems to be easier to rationalize than the method of equal additions, provided that the pupil is bright enough to understand the borrowing operation.

Some contend, on the other hand, that the method of equal additions is easier to learn than is the method of decomposition. This is particularly true, they say, when there are a number of successive zeros in the minuend. In the example shown, a rational explanation of how we borrow is long and involved. Each of the zeros is

$$\begin{array}{r} 40005 \\ 21738 \\ \hline 18267 \end{array}$$

[1] Ballard, P. B. "Norms of Performance in the Fundamental Processes of Arithmetic with Suggestions for Their Improvement." *Journal of Experimental Pedagogy,* II: 396-405 and III: 9-20, December 5, 1914, and March 5, 1915.

[2] McClelland, William W. "An Experimental Study of the Different Methods of Subtraction," *Journal of Experimental Pedagogy,* IV: 293-299, December 5, 1918.

[3] Winch, W. H. "'Equal Additions' *versus* 'Decomposition' in Teaching Subtraction: an Experimental Research," *Journal of Experimental Pedagogy,* V: 207-220, 261-270, June 5 and December 6, 1920.

changed into a ten and then into a nine before the operation of subtraction can proceed. The method of equal additions seems to be easier in such cases.

It would be very unwise to contend that either method is distinctly superior or inferior to the other. That teachers are successful with both methods, there can be no doubt, but adequate scientific evidence as to the relative merits of the two methods is not available. The author is inclined to favor the method of decomposition (take-away-borrowing) because it seems to be somewhat easier to rationalize and because it seems to be the more extensively used method. A survey of about one thousand of his own students indicated that approximately three-fourths had been taught this method while pupils in the grades, while about one-fourth had learned the method of equal additions. About one in a hundred had been taught the addition-carrying method. He has yet to find in his classes a student who has been taught the addition-borrowing procedure.

Pending the announcement of the results of wide scale experimentation, the best policy for the teacher seems to be a *laissez-faire* one; that is, do not undertake to change what the majority of the pupils in the school or of the people in the neighborhood are doing. The teacher had better change her habits than ask a large number of pupils to change theirs unless there is clearly evidence favoring the change. If the pupils are primary pupils who have not learned subtraction, the teacher should be guided by the prevailing customs in the community for it is hardly worth while to set up a state of confusion between school and community if no particular benefit is to be obtained.

Because the borrowing method seems to be more generally used than the carrying method the writer has adopted it for use in illustrative examples later in this chapter. The reader should not, however, conclude from this fact that the writer is urging the adoption of the borrowing method. Only favorable experimental evidence would make him do that.

The second dispute. The more recent of the two disputes is that as to whether subtraction should be taught as subtraction or as addition. Here again we find much discussion and little experimentation. Mead and Sears[4] reported some superiority for the "take-away" method. Taylor[5] found that, although children were taught the addition method, most of them eventually used another procedure. Beatty's[6] results were similar to those of Taylor.

Until recently, the argument has largely favored the addition method. Two reasons have been given. First, the addition method makes use of the addition combinations, meaning that the pupil has less to learn. Second, the addition method supposedly resembles making change. As to the second argument, in making change one does not ordinarily subtract at all. That is, he neither takes away the amount of the purchase from the amount paid, nor does he add to the amount of the purchase that which will make the amount paid, as in additive subtraction. Rather, he counts in terms of our monetary units. Thus, if a clerk is given a five-dollar bill in payment for a purchase amounting to $2.68, he will ordinarily say to himself as he picks the appropriate coins and bills from the till, or as he counts the change out to the customer, "268, 70, 75, 3, 4, 5." Practices followed in making change do not constitute an argument for either method of teaching subtraction.

The former argument—use of addition combinations—has more in its favor. It may quite well be easier for children to learn subtraction in this way but whether the gain is a permanent one is a more serious question. The argument, admittedly, is a theoretical one. As we shall see, however, there are considerations of a theoretical nature which support the "take-

[4] Mead, Cyrus D., and Sears, Isabel. "Additive Subtraction and Multiplicative Division Tested." *Journal of Educational Psychology*, VII: 261-270, May, 1916.

[5] Taylor, Joseph S. "Subtraction by the Addition Process." *Elementary School Journal*, XX: 203-207, November, 1919.

[6] Beatty, Willard W. "The Additive *versus* the Borrowing Method of Subtraction." *Elementary School Journal*, XXI: 198-200, November, 1920.

away" method quite as well as, or better than, the method of addition is supported by the argument that children thereby have less to learn.

As indicated in Chapter IV, we have adopted for illustrative purposes the "take-away" method rather than the additive method. We have arrived at this decision not because the "take-away" method is admittedly the better—the evidence is not yet in—but because it seems to be the most widely used method at the present time and because it seems that the preponderance of theoretical considerations favor it. Without entering into a detailed discussion of these considerations, we may simply state the principal ones which have led us to adopt this procedure.

In the first place, the take-away method represents what most people think when reference is made to subtraction. This is shown in the etymology of the word. "Subtraction" comes from the Latin, *subtractus,* a participial form of *subtraho,* meaning, literally, "draw from under." The same meaning stands today in ordinary affairs. Depreciation in the value of a house, or a car, is not thought of as an amount to be added to the present sale value. Contemplated expenditures are deducted from what we have, not added to what we shall have left. Reductions in marked-down sales represent what the customer saves out of the original price rather than what should be added to the present price to produce the list price. To be sure, some situations demand the "how-much-more" idea as in the case of the man who has $850 and figures how much more he needs to buy a car costing $1095, but even here, he is likely to determine the amount needed by "subtracting" $850 from $1095, rather than by thinking "what amount added to $850 will make $1095." The take-away method *is* subtraction; the addition method is not subtraction but is a substitute for subtraction.

In the second place, if the take-away method is employed, an example can be checked by adding the remainder to the subtrahend to produce the minuend. This valuable check is lost

if the addition method is used. In additive subtraction, the only check consists of adding again the subtrahend and remainder, in the same order or, better, in the reverse order.

In the third place, the addition method is an awkward one to follow. In the accompanying example, one thinks, "3 and what make 5? 2," and writes the 2. Later he may shortcut this to "3 and (looking at the 5) 2 make 5" and write the 2. In either event, the eye must move up the column and then down to write the answer digit. In take-away subtraction, on the other hand, one thinks, "5 take-away 3 are 2" and writes the 2, or "5 less 3, 2." The eye-movements (and the hand-movements which frequently accompany eye-movements) are, in the latter case, more economical.

$$\begin{array}{r} 6945 \\ 4783 \\ \hline 2162 \end{array}$$

In the fourth place, if we would use the addition method we must have our pupils learn 10 addition facts besides those given in Chapter IV. Two of these are needed in the accompanying example. In solving this example by the addition-carrying method, one thinks "6 and 6 are 12, 10 and 3 are 13, 10 and 4 are 14, 6 and 1 are 7." The combinations $10 + 3$ and $10 + 4$ would not be needed if subtraction were taught as a take-away process. Before one can subtract by the addition-carrying method, then, he must learn the ten higher decade addition combinations,

$$\begin{array}{r} 7432 \\ 5996 \\ \hline 1436 \end{array}$$

$$\begin{array}{rrrrrrrrrr} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\ \hline \end{array}$$

If the addition-borrowing method is used, these ten addition combinations are not needed. The combinations used are those given in Chapter IV.

Finally, as stated before, the take-away method is the method in most general use today. In one city school system known to the author, the pupils were carefully taught the addition-carrying method (the Austrian method) in the primary grades but when tested in the sixth grade more than three-

fourths of them were found to be using the take-away procedure. It seems reasonable to conclude that the change had been made because most of the people of the community used the take-away process. School practices should depart from community practices if a demonstrable gain is thereby accomplished, but, unless a real advantage is forfeited, school practices should be in harmony with the practices of the community in which the school is situated.

SUBTRACTION WITH BORROWING

Pupils' reactions complex. Winch gives the detailed responses of several pupils in subtraction examples and notes the extent to which they reproduce in exact detail the steps in the procedure originally taught them. We quote the responses of one pupil on three examples. In the first example, shown on the right, she proceeded as follows: [7]

(1) 9 from 6 you cannot, take 1 from the 7 next door leaves 6, 9 from 10 is 1 and 6 is 7.
(2) 2 from 6 leaves 4.
(3) 8 from 5 you cannot, take 1 next door, leaves 3, 8 from 10 leaves 2 and 5 are 7.
(4) 9 from 3 you cannot, go next door, take 1 leaves 1, 9 from 10 is 1 and 3 makes 4.
(5) 3 from 1 you cannot, go next door, take 1 from the 6 leaves 5, 3 from 10 is 7 and 1 makes 8.
(6) 2 from 6 leaves 4.

$$\begin{array}{r} 624{,}576 \\ \underline{139{,}829} \\ 484{,}747 \end{array}$$

In the next example the examiner told her to "work the next sum very quickly." She proceeded as follows: [8]

(1) 7 from 11 is 4.
(2) 8 from 12 is 4.
(3) 9 from 10 is 1.
(4) 0 from 1 is 1.

$$\begin{array}{r} 2{,}131 \\ \underline{987} \\ 1{,}144 \end{array}$$

Evidently, this pupil's elaborate language responses were made by force of habit and were quite unnecessary. At the mere suggestion of speed, she dropped them.

[7] Winch, W. H., *loc. cit.*, p. 209.
[8] *Idem.*

The complexity of this pupil's responses may be further emphasized by detailing her solution of an example containing many zeros in the minuend. The example is shown. She proceeded as follows: [9]

(1) 9 from 0 I can't, go next door I can't, go next door I can't, go next door I can't, go next door I can't, go next door, take 1, leaves 3, and that makes that (pointing to the 0 immediately to the right of the 4 in the minuend) 10, 9 from 0 I can't, go next door I can't, go next door I can't, go next door I can't, go next door, take 1 from the 10 leaves 9, and makes that one (pointing to the 0 in the second place from the 4) 10. 9 from 0 I can't, go next door I can't, go next door I can't, go next door, take 1 from the 10 leaves 9 and makes that (pointing to the third 0) a ten. 9 from 0 I can't, go next door I can't, go next door, take 1, leaves 9 and makes that (pointing to the fourth 0) a ten. 9 from 0 I can't, go next door, take 1, leaves that a 9 and makes this a 10, 9 from 10 leaves 1.

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400,000
     59
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399,941
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(2) 5 from 9 leaves 4.
(3) 0 from 9 leaves 9.
(4) 0 from 9 leaves 9.
(5) 0 from 9 leaves 9.
(6) 0 from 3 leaves 3.

Buswell and Judd, after quoting the cases which we have just examined, say: [10]

"Such cases as the foregoing furnish convincing evidence of the need for more frequent analyses of the actual mental operations of pupils in working with subtraction. Pupils cannot reach the desired goals of speed and accuracy while such roundabout methods of procedure persist. In many cases, these wasteful and uneconomical methods continue to be employed through the upper grades and, in all probability, through adult experience. In order to correct adequately these wasteful processes of the pupil, detailed analyses of the type described should constantly be made by the teacher."

[9] Winch, W. H., *loc. cit.*, p. 213.

[10] Buswell, Guy Thomas, and Judd, Charles Hubbard. *Summary of Educational Investigations Relating to Arithmetic.* University of Chicago Press, 1925, p. 73.

Pupils who are very slow in doing subtraction examples will usually be found to be following roundabout procedures. Such procedures consume time and are often responsible for errors. If a teacher in the intermediate or upper grades will sit down with a pupil and ask him to say aloud what he thinks when he subtracts, she will often be surprised at the persistence of time-consuming habits which were established in learning subtraction with borrowing in the primary grades. It is the responsibility of the primary teacher to help a pupil form subtraction habits which shall be effective and economical.

Should pupils be taught the "why" of borrowing? In order to avoid the time-consuming methods which have been described, some recommend that pupils simply be shown *how* to borrow, in as brief and simple fashion as possible, and that we should not undertake to teach them *why*. As we have suggested with reference to carrying in addition, however, it seems best to rationalize new processes for pupils, if the explanations are within their comprehension. Ability to perform the fundamental operations with skill is a worthy goal, but skill *with understanding* is more meaningful, more interesting, and more educative. The elaborate responses which pupils sometimes make are not so much due to the fact that they have been shown why we borrow as to the fact that we have failed to help them develop efficient habits immediately after the meaning of borrowing was explained.

How to teach borrowing. Explanations similar to those which have been used in explaining the meaning of carrying in addition can be employed to show how and why we borrow in subtraction. Suppose that we start with a problem similar to that which was used in introducing carrying in addition. "Mary had 32 cents and spent 18 cents for valentines. How much did she have left?" We write the amounts on the blackboard as shown and draw lines under the 18, the 1, and the 8. The teacher may say, "We can't take 8 cents from 2 cents (pointing to the numbers at the right) but we can change 1 of Mary's

32 cents.	This is the same as	3 dimes and	2 cents.
18 cents.	This is the same as	1 dime and	8 cents.
14 cents.		1 dime	4 cents.

dimes to cents and put that with the 2 cents, making 12 cents. Then 8 cents from 12 cents leaves 4 cents. Since we changed 1 of the 3 dimes to cents, there are only 2 dimes left. Then, 1 dime from 2 dimes leaves 1 dime. Then Mary had left 1 dime and 4 cents which is the same as 14 cents.

After another blackboard illustration or two of the same kind, the pupils will be given a few similar problems to solve for themselves. By this time the meaning of the operation of borrowing should be clear. *Then immediate steps should be taken to shorten the process.* The teacher, extending her former explanation, may say: "We can do any subtraction example in this way, but we do not need to stop and think about changing dimes to cents. Look at this example. Of course I cannot take 7 from 5, so I just think of 5 as 15 and of the 4 as 3. Then 7 from 15 leaves 8, and 1 from 3 leaves 2. Here is another example. I can take 3 from 5, leaving 2. Then I think of the 6 as 16 and of the 4 as 3, because I can't take 9 from 6. Then 9 from 16 leaves 7, and 2 from 3 leaves 1. If the lower number is smaller than the upper number, we subtract just as we have been subtracting. But if the lower number is larger than the upper number, we think of how the upper number would look with a 1 in front of it and remember that the next upper number is 1 smaller than it was. Then we subtract. Now look at this example."

$$\begin{array}{r} 45 \\ -17 \\ \hline 28 \end{array} \qquad \begin{array}{r} 465 \\ -293 \\ \hline 172 \end{array}$$

$$\begin{array}{r} 647 \\ -218 \\ \hline \end{array}$$

Teacher: "Can I take 8 from 7?"
Pupil: "No."
Teacher: "Then how do I think of the 7?"
Pupil: "Think of the 7 as 17."
Teacher: "And how do I think of the 4?"
Pupil: "Think of the 4 as 3."
Teacher: "Then 8 from 17 leaves how many?"
Pupil: "Nine."

Teacher: "What is the next step?"
Pupil: "1 from 3 leaves 2."
Teacher: "And the next step?"
Pupil: "2 from 6 leaves 4."

Provide at once an abundance of subtraction examples made up of two- and three-digit minuends and subtrahends and in which borrowing is involved in just one step. While the pupils are working on these examples, move about the room, helping those who are having difficulty and correcting the mistakes of pupils who have made errors. An occasional blackboard explanation of a point which has proved troublesome to many in the class may be necessary. Do not assign examples for home work where wrong habits may be acquired but have all practice, especially in the earlier stages of learning a new topic, carried on under the direct supervision of the teacher.

As the pupils gain proficiency, the word, "borrow," may be introduced and used. Some object to this term since it is not literally true that we borrow, but the point seems a trivial one, having little bearing on how well the pupil learns. Instead of saying, "Think of the 4 as 14 and of the 6 as 5," we may substitute, "Since we cannot take 7 from 4, we may borrow a ten from the 6 tens, leaving 5 tens and increasing the 4 to 14. Then 7 from 14 leaves 7 and 2 from 5 leaves 3."

$$\begin{array}{r} 64 \\ -27 \\ \hline 37 \end{array}$$

When the pupils seem to have mastered those examples in which borrowing occurs in but one step, they may be given examples having minuends and subtrahends of three digits each and in which one must borrow in both the first and the second steps. In these, there should, of course, be an occasional step in which borrowing is unnecessary and the pupils should be warned to be on the lookout for such steps.

If we would have pupils develop efficient habits in subtraction we must see to it that they have an abundance of practice on examples incorporating the various types of difficulty and that this practice is under close supervision. Not only check up on the

answers which pupils get but also on the processes they use in arriving at those answers. The most effective way to block wrong paths is to direct pupils specifically into right ones, but sometimes they will get started in wrong directions in spite of the teacher's efforts (due, frequently, to out-of-school influences) and in such cases it is necessary to work with pupils individually, thwarting wrong procedures and insisting that right ones be used. Successful teachers frequently find it necessary to sit down with pupils and listen to them as they say aloud what they think in doing examples, or to place examples on the blackboard and listen to the responses of individual pupils as they subtract aloud.

As rapidly as pupils grasp the meaning of subtraction and understand the nature of the steps involved, urge that they short-cut their processes and arrive more speedily at their answers. Don't urge speed to the point where it causes confusion but try assiduously to prevent the acquiring of the elaborate and roundabout methods which were quoted from Winch. In this example, it may at first be necessary to think, "I can't take 7 from 5, so I borrow a ten from the 7 tens, leaving 6 tens and making the 5, 15. Then 7 from 15 leaves 8 and 3 from 6 leaves 3." But eventually, this should be shortened to "7 from 15, 8; 3 from 6, 3," which is thought as rapidly as one can write the digits 8 and 3 for his answer.

$$\begin{array}{r} 75 \\ 37 \\ \hline 38 \end{array}$$

The use of crutches. We sometimes find teachers who try to facilitate the pupil's understanding of the borrowing process by writing a "1" before the minuend figure and crossing out the adjacent minuend figure from which the 1 was borrowed and writing the smaller minuend figure in its place. The work may then resemble the example shown on the left. This device may make it easier for the pupil to see what takes place when he borrows and, perhaps, does more good than harm as a means of explaining the meaning of subtraction in examples of this kind. Could we guarantee that its use would stop here, no one would seriously object to it. But

$$\begin{array}{r} 3 \\ \not{4}16 \\ 2\ 8 \\ \hline 1\ 8 \end{array}$$

children tend to lean on crutches whether we wish them to do so or not. Having discovered an easy way to do examples involving borrowing, they quite naturally choose that method rather than a harder one and use it until it becomes habitual. Since no one would endorse this crutch for permanent use, then, it seems best not to use it at all. Other things being equal, teachers should not teach pupils procedures which are soon to be forbidden, but should employ approved methods from the beginning. Primary teachers are particularly liable to this sort of error. They permit pupils to use questionable methods because, for a time, they seem to yield results as good as those secured from better methods, forgetting that in later grades, the pupil will be seriously hampered by the habits thus acquired.

The zero difficulties. So far, we have used examples having zeros in the subtrahend, but have had no zeros in the minuend, except the single combination, 0—0, which has been introduced in such examples as those in Set III (page 97). The most difficult subtraction examples are those having zeros in the minuends. It is well, then, to postpone work on this type until other subtraction difficulties have been mastered. Then teach pupils to handle the zero difficulties and give, at once, intensive practice on examples of this category.

Start with a simple example such as is supplied by this problem: Clifford made 60 cents last Saturday and spent 28 cents for a kite that evening. How much did he have left? "This is the way we find out.

60 cents. This is the same as 6 dimes and 0 cents.
28 cents. This is the same as 2 dimes and 8 cents.
32 cents.

We can't take 8 cents from 0 cents, of course, so we borrow a dime from the 6 dimes, leaving 5 dimes and change the dime to cents, making 10 cents. Then 8 cents from 10 cents leaves 2 cents and 2 dimes from 5 dimes leaves 3 dimes. So Clifford had left 3 dimes and 2 cents, or 32 cents.

"John had $2.04 and spent 28 cents for a kite. How much did *he* have left?

```
$2.04
  .28
-----
$1.76
```

This time, there are no dimes, so we borrow a dollar, leaving 1 dollar, change the dollar to 10 dimes, borrow one of these dimes, leaving 9 dimes, and making 14 cents. Then 8 cents from 14 cents leaves 6 cents, 2 dimes from 9 dimes leaves 7 dimes, and 0 dollars from 1 dollar leaves 1 dollar. So John had $1.76 left.

"Howard had exactly two dollars and he, too, bought a kite for 28 cents. How much did *he* have left? This time we borrow 1 dollar, leaving 1 dollar, change it to 10 dimes, borrow 1 dime, leaving 9 dimes, and change the 1 dime to 10 cents. Then 8 cents from 10 cents leaves 2 cents, 2 dimes from 9 dimes leaves 7 dimes and 0 dollars from 1 dollar leaves 1 dollar. So Howard had $1.72 left.

```
$2.00
  .28
-----
$1.72
```

"When we have two zeros together in the upper number, then, we think of the first zero as 10 and the other zero as 9 and take 1 from the third figure. Here is another problem: Mr. Simpson bought 300 pounds of fertilizer for his garden. That evening he used 62 pounds on a flower bed. How much did he have left? Think of the first 0 as 10, of the next 0 as 9, and of the 3 as 2. Then 2 from 10 leaves 8, 6 from 9 leaves 3 and 0 from 2 leaves 2.

```
300
 62
---
238
```

"Sometimes, we have more than two zeros together in the upper number, as in this problem: The Smith family started to drive to California, 3000 miles away. The first day, they drove 284 miles. How far did they then have to go? We think of the first 0 as 10, of both the other 0's as 9's, and of the 3 as 2. Then 4 from 10 leaves 6, 8 from 9 leaves 1, 2 from 9 leaves 7, and 0 from 2 leaves 2. So they still had 2716 miles to go."

```
3000
 284
----
2716
```

The preceding paragraphs will not be given as one continuous explanation, but after each new type of example is explained the pupils will be given practice on examples in that group. They will gradually learn to do the more difficult examples and will

finally have the most difficult which the course of study in the primary grades supplies.

Additional sets of practice examples. The teacher should have at hand several sets of practice examples providing drill on the various subtraction difficulties which have been enumerated. These should be carefully prepared so as to provide practice on each of the types of difficulty and on each of the 100 subtraction facts (including the 19 zero facts). Set V does just this. Set V provides for four subtractions in each of 25 examples, 100 subtractions in all. The numbers are so chosen that each of the 100 subtraction facts is practised and that borrowing and the zero difficulties are included.

PRACTICE EXAMPLES IN SUBTRACTION. SET V

The 100 Subtraction Facts

6418 1376	3581 2611	7094 2397	4752 3523	9530 3975	3008 1462	6347 5068	8743 7408	2657 743
5041 4052	7761 2484	5930 5041	8522 4039	7155 3457	5123 1064	8461 8298	7380 6717	6000 2179
8754 5016	8720 6892	7852 988	9870 5798	4002 1966	6402 3985	6200 523		

In the preceding pages, we have indicated several types of subtraction examples to be used while teaching subtraction with borrowing. We have spoken of varying the length of the minuends and the subtrahends; of examples in which there is borrowing in one step, two steps, etc.; and of all of these with and without the various forms of zero difficulties. There should be at least one set of practice examples prepared for each of these types. Set V, above, is a summary set and is intended to be used after practice on the various sets which have been suggested. Preparing

such sets is a rather tedious task but the examples can be saved and used with succeeding classes.

Checking solutions. To insure accurate results checking is indispensable. In all written work in subtraction, pupils should habitually check their answers. If the habit is formed in the primary grades we may hope that it will persist in the intermediate and upper grades.

$$\begin{array}{r} 4629 \\ 3874 \\ \hline 755 \\ \hline 4629 \end{array}$$

Adding the remainder or difference to the subtrahend to get the minuend is the easiest and perhaps the best check to use. The sum so secured may be written, as shown on the left, or the adding may be done without writing the sum. It is probably best to write the sum in the first examples of a new type and then check without writing the sum. Writing the sum should not become a habit.

$$\begin{array}{r} 4629 \\ 3874 \\ \hline 755 \\ 3874 \\ \hline 4629 \end{array}$$

Some teachers have checking done by copying the subtrahend beneath the remainder and then adding, as shown. This adds somewhat to the labor involved but avoids the difficulty of adding across the line, a feature which pupils are not accustomed to. The chief disadvantage in this method, however, is the fact that pupils who have used it do not readily make the transition to checking without writing the sum, since the operation of checking without writing the sum is considerably different from the checking method which they have been using.

Summary. In elementary work in subtraction, primary teachers should strive to develop in pupils the ability to find differences and remainders with fair speed and with a high degree of accuracy. We may summarize the principal steps in the teaching process, briefly, as follows:

1. Explain new procedures and new operations well. Make sure that the pupil understands the procedure and that he can successfully follow the steps in the operation. Some teachers make their explanations too brief and do not use a sufficiently large number of illustrative examples. The illustrations and

explanations which have been given in the preceding pages are intended to suggest the method to be employed; they are probably too brief for most classes. Naturally, the amount of explanation and the number of illustrations necessary will depend upon the ability of the class and the adequacy of their previous training.

2. Recognize the various difficulties in subtraction examples and attack these difficulties one at a time. More and more, primary teachers are realizing the importance of making a detailed scientific analysis of the materials of the curriculum and the necessity of arranging a graded list of exercises through the use of which the pupils may gain a thorough knowledge of the subject-matter and acquire the desired habits.

3. Give an abundance of practice material. Keep in mind the Law of Exercise which says that, other things being equal, use strengthens and disuse weakens mental connections. The amount of practice should be rather large at the time of the first presentation and immediately thereafter, but may then gradually become less and less. The intervals between practice periods should be brief at first and then gradually become greater. In other words, the teacher should be acquainted with the best methods of providing and distributing practice on the fundamental bonds.

4. In providing for practice, see that no fundamental bonds are neglected. Ordinarily, the more recently presented and the more difficult items of subject matter will receive the major emphasis in sets of practice exercises, but sometimes it is well to arrange a summary or review practice set in which all of the skills so far taught in a given operation are reviewed. Set V, in this chapter, is an example. The haphazard methods of many teachers who provide just any examples for practice purposes should be displaced by more careful, scientific methods which see to it that each fundamental bond is given an adequate amount of exercise.

5. The close relationship of interest to effort is universally recognized. The Law of Effect tells us that the progress which

pupils make depends upon the extent to which they enjoy their work. See to it, then, that the pupil's introduction to a new difficulty is accompanied by a situation which appeals to him. An interesting problem, the solution of which demands the skill which is to be acquired, should usually precede actual instruction in a new type of example. The game element should be prominent in the drill devices used. Occasional progress tests to show the pupil where he stands are helpful.

If pupils see good reasons for mastering their subject matter difficulties, if they enjoy the games and other drill devices which are used, and if the lessons are properly organized and well taught, progress will be rapid and sure.

The use of problems. Problems involving the operation of subtraction should be freely used. The only purpose in learning subtraction is to develop the skill necessary to handle problems which involve this operation. There is danger that the drill work will become uninteresting to the pupils if they do not have frequent occasion to apply the skills developed to problems. In choosing and formulating problems, remember that they should be real, they should make a strong appeal to the imaginations of the children, and they should be worded in language which the children can readily understand.

The question of higher decade subtraction. Many writers on the teaching of arithmetic recommend that after pupils have learned the subtraction facts and before they are taught to do subtraction examples, they should be instructed in higher decade subtraction. Higher decade subtraction is, of course, analogous to higher decade addition. Just as pupils have extended their knowledge of the addition facts to higher decades, with and without bridging tens, now they may similarly extend their knowledge of the subtraction facts. There would be two groups of these higher decade combinations: (1) those in which the remainder is in the same decade as the minuend; and (2) those in which the remainder is in the decade next below that of the minuend.

The case of higher decade subtraction is quite different from

that of higher decade addition, however. In a large proportion of our addition examples we need skill in higher decade addition; indeed, we cannot well get along without it. But in no subtraction examples do we need higher decade subtraction, except, possibly, in examples like $\begin{array}{r}27\\ -\underline{18}\end{array}$. But such a small proportion of our subtraction examples is made up of examples like this, that this hardly seems a sufficient justification for the time and energy required to learn higher decade subtraction.

Higher decade subtraction is required in short division, however. In the example shown one must subtract 35 from 41 before the second step in division can be taken. But, it is pointed out in Chapter VIII that short division will not be taught in the primary grades (Grades I–III). Higher decade subtraction should be presented immediately preceding instruction in short division, which will probably be in the 4A grade.

$$\begin{array}{r}7\overline{)4165}\\ 595\end{array}$$

EXERCISES

1. Why are the subtraction habits of pupils frequently better than their addition habits?

2. Give examples of unusual procedures which you have known pupils to employ in subtraction.

3. What principles governed the making of the sets of practice examples shown in this chapter?

4. What means is taken to bridge the gap from subtraction examples without borrowing to examples with borrowing?

5. Illustrate by an example each of the four methods of subtraction.

6. Which method were you taught? Do others seem awkward to you?

7. What would make you decide which method to use if you were to teach in a new community and had your choice of methods?

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Chapter VII

TEACHING THE MULTIPLICATION AND DIVISION FACTS

The reader will recall that in Chapter IV we discussed the teaching of the addition and subtraction facts and that it was suggested that the work be organized into teaching units in such a way that the two processes should be taught together. In like manner, we would teach the multiplication and division facts together. This procedure seems to be the economical one to follow.

The terms used. In this chapter, we shall use the terms *multiplication combination, multiplication fact, division combination, division fact,* and *teaching unit* in exactly the same way that we used similar terms with reference to addition and subtraction. For example, the two digits, 4 and 6, give us a teaching unit composed of a multiplication combination and a division combination. For the multiplication combination there are the two multiplication facts, 4×6 and 6×4; for the division combination, there are the two division facts, $24 \div 4$ and $24 \div 6$. Thus, for each teaching unit there are four primary facts, except when the two digits are the same as, 5 and 5, in which case there are but two facts—the multiplication fact, 5×5, and the corresponding division fact, $25 \div 5$.

Writing the combinations. In Chapter IV, all the addition and subtraction combinations were written in the vertical form rather than in the equation form. After children have thoroughly learned their combinations in the vertical form, it is well to use the equation form also in drills and to give them practice in reading the signs, + and —. But since we do our addition and

subtraction examples in the vertical form rather than in the equation form, it seems wasteful to use the latter excessively in the training of pupils.

A similar situation exists in multiplication and division. If a boy is to find the cost of suits for a ball team at $5.75 each, he writes $\begin{array}{r}\$5.75\\ \underline{9}\end{array}$ and not 9 × $5.75. Or, if 4 boys plan to buy a motor boat costing $82.00, they find the amount which each must pay by solving the example in this form, $4\overline{)\$82.00}$, not $82.00 ÷ 4. In spite of the fact that we rarely use the equation form in later work in multiplication and division, many teachers conduct their drills lessons on the multiplication and division facts almost exclusively by expressing these facts in the equation form. In this chapter, we shall write multiplication combinations in the form $\begin{array}{r}7\\ \underline{\times\ 6}\end{array}$, not 6 × 7, using the "×" for a while to remind the pupil that he is not adding or subtracting, and our division combinations in the form $7\overline{)42}$, not 42 ÷ 7.

The zero combinations. Studies of the difficulty of the combinations show that zeros cause much trouble. A pupil, in multiplying 6 by 0, or 0 by 6, is quite likely to write 6, instead of 0, for his product. Both the results of scientific studies and the testimony of classroom teachers indicate that very specific provision must be made for teaching the zero difficulties in multiplication and division, as in addition and subtraction.

We did not, however, include the zero combinations with the other addition and subtraction combinations. We left them for later treatment for two reasons: (1) "the zero combinations as separate combinations do not occur in the pupil's real number experiences," (2) "learning the zero combinations is not closely analogous to learning the other . . . combinations" (pages 29-30). For the same two reasons, we shall not include the zero combinations with the other combinations in multiplication and division. Pupils do have occasion to find the cost of 3 rubber balloons at 5 cents each but they do not have occasion to find the cost of

0 rubber balloons at that price. Likewise, 3 boys who have been nutting may have occasion to divide 15 capfuls of hickory nuts by 3, but if the quest for nuts was unsuccessful, they do not have occasion to ponder over the amount which each would receive if 0 capfuls are divided among the 3 boys. Efforts to formulate problems involving the zero combinations in multiplication and division are likely to result in verbal statements which confuse children or constitute an insult to their intelligence.

The zero combinations should be taught in connection with the situations in which they occur, in examples like $\begin{array}{r}20\\ \underline{\times 3}\end{array}$, or $2\overline{)40}$. That is, they will be taught when the pupils are learning to do examples in multiplication and division, some time after the primary facts have been learned. Then we shall teach all zero combinations in multiplication as a group by teaching the principles that *any number times zero equals zero* and that *zero times any number equals zero*. Likewise, the principle that *zero divided by any number equals zero* will cover all of the cases of dividing zero by a number. Later, when the need for it arises, pupils will be told that it is impossible to divide by zero.

There is some evidence to the effect that with proper methods of teaching the zero combinations are no more difficult than are the other combinations. Fowlkes reports that there is reason to doubt the validity of the emphasis which has been placed upon the difficulty of the combinations involving zero.[1] It may quite well be that the many errors which pupils make when operating with zeros are due to insufficient opportunity to practise with zeros under proper learning conditions.

The number of combinations. The number of combinations will, of course, depend upon how far the pupil goes with the "tables." A few years ago, there were practically no schools in which the pupils did not learn their tables from 1×1 to 12×12, making 78 combinations, or 144 facts. In a few schools,

[1] Fowlkes, John Guy. "A Report of a Controlled Study of the Learning of Multiplication by Third-Grade Children." *Journal of Educational Research*, XV: 181-189, March, 1927.

years ago, pupils were required to learn tables to 15 × 15 (120 combinations, or 225 facts) and, rarely, to 20 × 20 (210 combinations, or 400 facts). It was finally recognized, however, that learning the combinations beyond 9 × 9 added little, if anything, to a pupil's skill in multiplying but added so much to his learning difficulties that the primary facts up to 9 × 9 were much less likely to be mastered.

After all, does it not seem strange that we should ever have required pupils to learn multiplication tables beyond 9 × 9? We were wont to argue that the practice of learning the tables to 12 × 12 would save the pupil much time for he would do examples whose divisors were 11 and 12 by short division and in multiplication examples having 11 or 12 as multiplier he would write out the products at once instead of writing two partial products and then adding them. As a matter of fact, however, the great majority of adults who have been so trained actually use the long division form when dividing by 11 or 12 and also employ the longer form in multiplying by 11 or 12. Smith[2] states that the custom of extending the tables to 12 × 12 can be traced to the early influence of English educational practices. In that country the fact that there are 12 pence in a shilling made it seem advisable that the 12's be taught. The 11's were put in for logical reasons—to make the system complete.

Later, after the pupil has become proficient in elementary work in multiplication, a few of the 12's should be learned because of their use in connection with things in dozens and inches in feet—say, up to 6 × 12. Also, a few of the 15's should be learned because of the large number of articles which sell at 15¢ each, 15¢ a pound, 15¢ a dozen, 15¢ a can, etc. But in the pupil's first work in multiplication—the work which is ordinarily done in the third grade—instruction will be limited to the combinations up to 9 × 9 with later provision for zero difficulties.

Limiting the combinations to 9 × 9 means that we have but

[2] Smith, David Eugene. *The Progress of Arithmetic in the Last Quarter of a Century.* Ginn and Company, 1923, pp. 43-44.

45 teaching units. But if we go to 12×12, we have 78 teaching units. The task of learning the multiplication and division facts is considerably reduced and much simplified, then, if we teach only to 9×9.

The writer's observations and the results of the many tests which he has given to children in the third and fourth grades force him to believe that, in general, the teachers in these grades are falling far short of what they should attain in teaching the primary facts of multiplication and division. Most courses of study provide that the multiplication facts be learned before the pupil leaves the third grade. But many third grade pupils, perhaps most of them, are promoted to grade four with very inadequate skills in these facts. *Third grade teachers are failing to bring their pupils to a satisfactory standard of attainment.* The reason is probably two-fold: first, the common inclusion of the tens, elevens, and twelves, particularly the elevens and twelves, making 78 teaching units instead of 45; second, poor methods of teaching, particularly serial memorizing of tables. The recommendations of this chapter are designed to aid the teacher in bringing her pupils to a *mastery* of those primary facts in multiplication (and division) which they need.

Taking the combinations to 9×9, there are, of course, exactly as many multiplication combinations as addition combinations, as many multiplication facts as addition facts, and as many teaching units in multiplication and division as in addition and subtraction. In other words, we have 45 multiplication combinations and 81 multiplication facts, 45 division combinations and 81 division facts, and the entire 162 facts are organized into 45 teaching units. The 45 combinations which will form the 45 teaching units are listed on pages 27-28.

Serial memorizing of tables. Many years ago it was a common practice to require pupils to memorize tables in each of the four fundamental operations—addition, subtraction, multiplication, and division. At the present time, however, very few, if any, American schools have pupils memorize tables in addition,

subtraction, or division. But the practice of committing to memory the multiplication tables in serial order is still quite common. Pupils recite their ones, their twos, their threes, and so on to the twelves.

The more progressive schools have abandoned this procedure. They recognize that when a pupil has learned his sixes, he cannot answer the question, "How many are 4 sixes?" without implicitly reciting, "1 six is six, 2 sixes are twelve, 3 sixes are eighteen, 4 sixes are twenty-four." This delayed response is not adequate for such multiplication examples as the one shown. Here, as in addition, we want the connection between "four times six" and "twenty-four" to be *direct* and the response *immediate*. Pupils who have memorized their multiplication tables in serial order must, after all, learn the combinations anew before they can solve examples effectively. This they are much less likely to do since they already possess a roundabout method by which they can more easily obtain the answers.

$$\begin{array}{r} 426 \\ \underline{4} \end{array}$$

The order of teaching the combinations. Reciting tables to find a given product in multiplication is analogous to counting to find a sum in addition. Indeed, reciting multiplication tables *is* counting. The pupil who "says his eights" to find how many 6 eights are, actually counts by eights to 48. Some contend that the best way to train a pupil to give products automatically is to have him learn to recite his tables in serial order. Experimental evidence as to the best method for teaching the multiplication combinations has not been produced, but it seems to be better theory to teach each combination specifically and to train the pupil to give each product directly without running through a circuitous train of associations. At any rate that is the kind of skill which he must eventually come to possess if he is to solve ordinary multiplication examples effectively.

The two principles stated on pages 32 and 33 for deciding the order in which the teaching units of addition and subtraction are presented are also applicable to the teaching units of multiplica-

tion and division. They are: (1) present the combinations in an order which will not encourage children to count; (2) arrange the combinations approximately in the order of their difficulty. According to the first principle, we shall not present consecutively two combinations having a common factor with the other two factors consecutive numbers; that is, $\begin{array}{r}4\\ \times 5\\ \hline\end{array}$ will not be given immediately preceding or following $\begin{array}{r}4\\ \times 4\\ \hline\end{array}$ or $\begin{array}{r}4\\ \times 6\\ \hline\end{array}$, etc. To put the second principle into effect, we shall turn to Clapp's lists of combinations arranged in the order of their difficulty.[3]

We get, of course, four distinct difficulty lists, two for multiplication and two for division. There will be one difficulty list for the multiplication combinations stated in one order and another list for combinations stated in the reverse order. The same is true for the division combinations. In preparing the list which follows, the 45 combinations were each given four rankings—one for each list—and the four rankings were averaged to determine the position which a combination, or teaching unit, should occupy.

This final difficulty ranking is not highly reliable. It is unreliable because the position which a combination occupied sometimes differed greatly in the four lists. Furthermore, the difficulty of these combinations was determined by the mistakes which children made after the combinations had, presumably, been learned. The learning difficulty may be quite different. Also, better teaching methods may reduce greatly the difficulty of some combinations. For example, the most difficult combination in the list, as finally arranged was $\begin{array}{r}1\\ 1\\ \hline\end{array}$. Perhaps the meaning of $\begin{array}{r}1\\ \times 1\\ \hline\end{array}$ and of $1\overline{)1}$ had never been made clear to these pupils. For these reasons and because we wish to introduce combinations having small numbers early in order that a greater number of examples

[3] Clapp, Frank L. *The Number Combinations: Their Relative Difficulty and the Frequency of Their Appearance in Text-Books.* Bureau of Educational Research, Bulletin No. 2, Madison, Wisconsin: University of Wisconsin, 1924, p. 5.

without carrying may be solved while pupils are still learning the combinations, a few have been moved a considerable distance from their original positions. It is believed that the list as finally arranged can be defended as successfully as can any other arrangement, in the light of our present knowledge of the teaching of the multiplication and division facts. The list, divided into five groups of nine teaching units each, is given below.

2	5	8	4	6	3	7	9	3
2	2	2	2	2	2	2	2	3
—	—	—	—	—	—	—	—	—
4	5	4	5	6	6	5	7	9
4	5	3	4	3	5	3	5	4
—	—	—	—	—	—	—	—	—
5	2	4	7	3	8	6	9	1
1	1	1	1	1	1	1	1	1
—	—	—	—	—	—	—	—	—
8	7	8	6	7	8	9	9	7
5	3	4	4	6	3	5	3	4
—	—	—	—	—	—	—	—	—
9	7	8	9	8	9	6	8	9
6	7	6	7	7	8	6	8	9
—	—	—	—	—	—	—	—	—

HOW TO TEACH THE COMBINATIONS

Review the principles of habit formation. Before launching upon the teaching of the primary facts of multiplication and division, the teacher should study again the principles of habit formation as summarized on pages 35 and 36 in connection with our discussion of the teaching of the primary facts of addition and subtraction. The analogy between the teaching of the fundamental facts of addition and subtraction and those of multiplication and division is quite close. What we desire in each

case is a direct connection between stimulus and response, a bond that shall act immediately and correctly. When the pupil sees or hears $\begin{array}{r}9\\ \underline{\times 5}\end{array}$ we expect him to react to this stimulus with the response, "45," and to react without following any such indirect procedure as, "one times nine is nine, two times nine is eighteen, three times nine is twenty-seven, four times nine is thirty-six, five times nine is forty-five."

The relationship to addition. Multiplication is a short method for addition. That is, when I multiply 37 by 8, I am really employing a short method for adding 37 eight times. When we have a large number of addends, *all the same size,* we employ multiplication instead of addition to save time and to reduce the chances for error. If the addends are different, we, of course, combine them by addition.

It is well that pupils should learn that multiplication is a short cut for addition. There are two reasons: first, it helps to show the pupil the *meaning* of multiplication; second, it gives him a *motive* for learning multiplication. If multiplication as a process is presented independent of its uses and without making clear to the pupil the advantage which will accrue to him from learning it, there is danger that his progress in learning will be slow and unsatisfactory. To teach well, we must take full advantage of the Law of Effect.

Our arrangement of the combinations makes it possible to take excellent advantage of this relationship to addition. It will be seen that we have used the smaller numbers as multipliers, have in general given the smaller multipliers first, and have begun the list with the twos. Using the smaller number as multiplier makes it easier to develop a combination by addition; it is easier, for example, to show by addition $\begin{array}{r}9\\ \times 2\\ \hline 18\end{array}$ than to show $\begin{array}{r}2\\ \times 9\\ \hline 18\end{array}$. For the same reason, the smaller multipliers occur early in the list of combinations and the larger multipliers, later.

There are two advantages in beginning with the twos. In the first place, Clapp's study shows that these are among the easiest of the combinations. In the second place, the relationship to addition is so close, that the twos are quite easy to teach. The twos in multiplication are simply the doubles in the addition combinations stated differently. If the pupil has learned that 5 and 5 are 10, he has learned that two 5's are 10, but has not yet learned to say it that way. If we begin with the twos, it is relatively easy for the pupil to see the relationship of multiplication to addition.

Some will contend that we should carry this argument still further and begin with the ones. If we begin with the ones, however, it is very difficult to show the pupil that he is multiplying. Indeed, he is *not* multiplying, in the literal sense of the term. To look at a normal third grade boy and say with a straight face, "How much is one times 6?" is likely to either baffle him or insult his intelligence. One might as well say, "How many is one dog?" as "How many is one six?" When we take up combinations having 1 as the multiplier, then, we shall dispose of them in rapid fashion. We merely need to show that one 4 and four 1's mean the same thing, 4 in each case, and give sufficient practice to fix the bond so that it will act surely and readily when needed. It seems much better to begin with twos than with ones if we would teach the meaning of multiplication and show effectively the relationship between multiplication and addition.

Teaching the twos. As has already been indicated, the twos are easy to teach because of their intimate relationship to the doubles in addition. Pupils who are proficient in their addition combinations—and many pupils are proficient in the doubles who do not know the other combinations well—already know the twos in multiplication and need, merely, to learn the *meaning* of multiplication and to become accustomed to the *language* of multiplication.

For the first two combinations, $\begin{array}{r}2\\ \times 2\\ \hline\end{array}$ and $\begin{array}{r}5\\ \times 2\\ \hline\end{array}$, the lesson may proceed as follows:

Teacher: "Let us go back to some of our numbers in addition. How many are 2 and 2?"

Pupil: "Four."

Teacher: (Writing on the blackboard, $\begin{array}{r}2\\ 2\\ \hline 4\end{array}$) "Yes. Two and two are four. How many 2's did I write?"

Pupil: "Two."

Teacher: "Then we can say that in another way. Instead of saying, 'two and two are four,' we can say 'two 2's are four.' Now let us take another example. How many are 5 and 5?"

Pupil: "Ten."

Teacher: (Writing $\begin{array}{r}5\\ 5\\ \hline 10\end{array}$) "Yes. Five and five are ten. How many 5's did I write?"

Pupil: "Two."

Teacher: "In what other way can I say that five and five are ten?"

Pupil: "You can say, 'two 5's are ten.'"

Teacher: "When we want to say, 'two 5's are ten,' instead of writing it as I have written it here, we write it this way $\begin{array}{r}5\\ \times 2\\ \hline 10\end{array}$. That means, two 5's are 10, or, 2 times 5 are 10. The sign in front of the 2, '×', means 'times.' When I write this, $\begin{array}{r}2\\ \times 2\\ \hline 4\end{array}$, I mean, two 2's are four, or, 2 times 2 are 4."

It will be necessary to repeat this explanation for some of the pupils, and, perhaps, give portions of it in modified form a third time. While the detailed explanation is being made, one pupil after another is given the opportunity to think and say, "two 2's are four" (or 2 times 2 are 4) and "two 5's are ten" (or 2 times 5 are 10). Remember that the purpose here is not to give new arithmetical facts but to give old facts in new form and to acquaint the pupils in a limited way with the language and meaning of multiplication.

It may be well to present the third combination, 2 × 8, in the same lesson. This time we abbreviate somewhat the former explanation. After writing on the blackboard the addition combination, 8 + 8, we call upon the pupils to state and write the other way in which this can be expressed. With the three combinations before them and with considerable practice in thinking, writing, and saying the new language, the pupils will have made a good beginning in learning their multiplication facts. The work so far given can probably be accomplished in a single lesson.

Presenting the related facts. The next day, after reviewing the work presented the preceding day, the teacher may proceed to the related multiplication facts—five 2's rather than two 5's, and eight 2's rather than two 8's. Taking the former as a specific example, we write on the blackboard both two 5's and five 2's, and by actual addition show that each gives us a sum of 10. Two 8's and eight 2's are similarly treated. Having shown the children that 2 times 5 means the same answer as 5 times 2, and that 2 times 8 is the same as 8 times 2, we generalize to the effect that no matter whether we write

$$\begin{array}{r} 5 \\ \underline{5} \\ 10 \end{array} \qquad \begin{array}{r} 2 \\ 2 \\ 2 \\ 2 \\ \underline{2} \\ 10 \end{array}$$

$\begin{array}{r} 5 \\ \underline{\times 2} \end{array}$, or $\begin{array}{r} 2 \\ \underline{\times 5} \end{array}$, etc., the answer is the same in either case, that is, no matter which of the two numbers is written below with the sign "×" before it, we get the same answer. This generalization will be reënforced and strengthened by presenting several later combinations in the two addition forms, but it will not be necessary to use the two forms in presenting the entire 45. We should, however, present the two forms for at least half of the combinations, or until 2, 3, 4, and 5, have occurred as multipliers.

The corresponding division facts. After these five multiplication facts (three combinations) have been presented, the teacher should proceed at once to the corresponding division facts. The pupil has learned that two 2's are four, that two 5's are ten, that five 2's are ten, that two 8's are sixteen, and that eight 2's are sixteen. The lesson will then take this form.

Teacher: "How many 2's in four?"
Pupil: "There are two 2's in four."
Teacher: "How many 2's in ten?"
Pupil: "There are five 2's in ten."
Teacher: "How many 5's in ten?"
Pupil: "There are two 5's in ten." And so on.

The exact language employed will depend upon the language which has been used in connection with the multiplication facts. If we have been saying, "Two 5's are ten," we shall now say, "How many 5's in ten?" and "There are two 5's in ten." If, on the other hand, we have been saying, "2 times 5 are 10," we shall now probably say, "2 into 10, how many times?" and "2 into 10, 5 times." Some prefer the one form, some prefer the other, and some may prefer neither. We do not know which language forms are best for teaching purposes, but we do know that the language we use should be simple language which children understand, and that, in the pupils' early experiences with a new topic, we need to be consistent in the words and expressions which we employ.

The written forms which the pupil is first taught should be those which he will have the greatest need for in later work, other things being equal. This will bar the very early use of the equation forms of statement with the signs, "$\times$" and "$\div$." Rather than write $2 \times 5 = 10$ and $10 \div 2 = 5$, we shall, for a time, write $\begin{array}{r}5\\ \underline{\times 2}\end{array}$ and $\begin{array}{r}5\\ 2\overline{)10}\end{array}$. The sign, "$\times$," before the multiplier is put there temporarily to help the pupil remember that he is not adding or subtracting and its use will soon be discontinued. Later, as a means of adding variety to the drill lessons, the equation forms will be used and the pupils made familiar with the signs but, as was pointed out in a previous page, the early emphasis should be upon the written forms which have been recommended.

Teach the four facts together. In presenting the first three teaching units, the division facts were neglected until the multiplication facts had been learned. This was done in order that

the pupil might have an opportunity to get a fair grasp of the meaning and language of multiplication. Then, in a subsequent lesson, the pupil's entire time was given to the corresponding division facts. In presenting the fourth and later teaching units, however, all four of the primary facts (two in the case of the doubles) will be taught in a single lesson.

The larger combinations. We have pointed out the relationship of multiplication to addition and have suggested that multiplication be taught as a case of abbreviated addition. That is, each multiplication combination is first presented as a case of column addition and then provision is made for permanently retaining the multiplication facts, and the corresponding division facts, in mind.

If the recommendations of Chapter V on higher decade addition have been carried into effect, these pupils have received training on all higher decade addition combinations whose sums are not greater than 39. This will be an adequate basis for the teaching of the majority of the multiplication combinations, but not for those whose products are equal to, or greater than, 40. For these, we shall be obliged to extend our former lessons in higher decade addition. As an alternative plan, we can simply *tell* the pupils the products of these larger combinations. Some may prefer to follow this plan, but it seems better to *show* the pupils what the product is in each case. The extra higher decade addition combinations needed should be easily learned, particularly if the 225 already learned have been taught as an extension of the primary facts. In a situation like this, we may expect considerable transfer. I do not know, for example, that I have ever added 1347 and 9, but when I think of these two numbers, I instantly think of their sum, 1356. I have had such extensive practice in adding 9 to a number of two or more digits ending in 7, that the skill acquired readily transfers to any other similar combination which I may encounter.

There are 11 multiplication combinations whose products are

greater than 39. Stated in the order in which they occur in the list on page 127, they are:

8	7	9	9	7	8	9	8	9	8	9
5	6	5	6	7	6	7	7	8	8	9

The additional higher decade addition combinations needed to teach these 11 multiplication combinations by addition are:

35	42	32	40	48	56	36	45	54	63	72
7	7	8	8	8	8	9	9	9	9	9

Of these 11 higher decade additions, five have sums in the forties, two have sums in the fifties, two in the sixties, one in the seventies, and one in the eighties. The seven whose sums are in the forties and the fifties may be of such value in long column addition as to justify their inclusion; at any rate, they do not constitute a very great extension of the addition abilities already acquired. There may be some question as to the value of the remaining four, aside from the purpose for which they are included, but it will be granted that our plan of teaching *all* the multiplication combinations by addition does not necessitate the learning of a large number of higher decade addition combinations which are of little value in themselves. These few will be developed as extensions of similar combinations in lower decades and will not be given extensive practice.

Let us illustrate the teaching procedure for these 11 combinations by reference to the third one encountered, $\begin{array}{r}9\\ \times 5\\ \hline\end{array}$ The teacher will first place on the blackboard the combinations, $\begin{array}{r}6\\ 9\\ \hline\end{array}$, $\begin{array}{r}16\\ 9\\ \hline\end{array}$, $\begin{array}{r}26\\ 9\\ \hline\end{array}$, $\begin{array}{r}36\\ 9\\ \hline\end{array}$. The pupils will review the first three combinations and get the sum of the fourth as an extension of these. In this setting, it should not be at all difficult for them to see that 36 and 9 are 45. If the three combinations have been well learned, and the ap-

proach to $\begin{array}{r}36\\ \underline{9}\end{array}$ through the review is well arranged, most of the pupils will no doubt see that 45 is the sum without being told. Then the five 9's will be written in the usual column form, added, and the sum, 45, written below. The teacher will then follow the procedure used in previous examples, writing, and practising on, $\begin{array}{r}9\\ \underline{\times 5}\end{array}$, $\begin{array}{r}5\\ \underline{\times 9}\end{array}$, $9\overline{)45}$, and $5\overline{)45}$.

$$\begin{array}{r}9\\9\\9\\9\\ \underline{9}\\45\end{array}$$

Providing for continued practice. Many of the statements made in Chapter IV with regard to the teaching units of addition and subtraction apply equally well to the teaching units of multiplication and division (see pages 41-48). In general, the teacher must go slowly and make provision for an abundance of practice on the facts of a teaching unit immediately after it is presented, if she expects to have these facts well learned. Continued practice must then be provided if the facts are to be retained but the amount of practice at successive repetitions may gradually become less and the intervals between practice periods may gradually become greater.

A great many games and devices have been prepared for drill on the fundamental facts of multiplication and division, particularly multiplication. Some of these are good, some are of doubtful value, and others are poor. To rate high a drill device must possess the following characteristics.

First, it must be interesting to the pupils for whom it is prepared. One of the most discouraging sights in a primary classroom is that of an unenthusiastic teacher conducting, in a wooden way, a drill lesson with a group of equally unenthusiastic pupils. The writer has seen such a lesson dragged out through a period of 20 minutes, while the pupils shifted restlessly in their seats, looked at each other and out the window, and sometimes had to be called upon twice before they would give attention to the part which had been assigned them. We must have repetition but it must be attentive repetition; not only the Law of Exercise but

also the Law of Effect must be in full operation, if results are to be worth while. A good drill lesson is brief and vigorous and the participants are eager and alert. The teacher must take these facts into serious consideration in selecting games and other drill devices.

Second, the game or device must provide much practice on the facts which are in need of drill. There are games which children enjoy playing and which are quite harmless as games, but which are nearly worthless as a means of teaching. Consider, for example, the following game.

Ring Toss Game

> A board 20 inches square, made of soft wood, is ruled into 25 smaller squares. Into each smaller square a small cup hook is screwed. Numbers, cut from the calendar, are pasted below the hooks. The pupils, one at a time, toss rubber rings (ordinary fruit jar rings) at the board. If the ring becomes caught on a hook, the pupil must give the product of the number of the hook by some number previously agreed upon or a number which the teacher, at that instant, calls. Each gets four tosses and scores one for each combination successfully answered.

In this game, only one pupil participates at a time, and the period of his participation is so long that not nearly all the class can take part in a single period. Success, too, depends too much upon deftness in tossing rings. Pupils scoring low may make low scores because of inadequate skill in ring tossing rather than because of insufficient knowledge of the combinations. This game, like many others, is a peculiar combination of two types of activity quite different, the sort of combination which is not likely to occur, except in a schoolroom. When men pitch quoits or horseshoes, they add points to get scores, but they do not, after a successful pitch, stop and say, "nine 8's are 72," or, "the sine of an angle of 30 degrees is ½," or recite the probable equation of the path traversed by the quoit or horseshoe.

In the third place, devices should provide for the active participation of a number of pupils, the more the better. If but one pupil can actively participate at a time, his period should be very brief and the active part should be passed to others in rapid succession.

Fourth, the device should conform to the recognized principles of learning as applied to drill lessons. This means that there should be an adequate amount of practice and that this practice should be properly distributed. It also means that only immediate responses should be acceptable; there should not be time to discover the answer to a combination by reciting the tables or counting, if it is not known at once, as is the case in the game, *Cat and Mouse.*

Cat and Mouse

> The mice form a circle with the cat outside. Each mouse holds a card containing a combination. The cat taps a mouse on the head and the mouse gives his combination. If he makes a mistake he is caught and must give his place to the cat.

Here each pupil has but one combination to concern him. If he does not know this combination he has an excellent opportunity to get his answer by unapproved methods. The lesson is on a low plane indeed when the teacher uses such a game as this to drill on the fundamental combinations!

For practising the fundamental combinations in multiplication and division, flash cards are of great value. Just as there were 162 cards in addition and subtraction, there will be 162 in multiplication and division. Of these, 81 will give the multiplication facts and 81, the division facts. Each card will be printed on both sides, one side with the answer and the other side without the answer. For example, a multiplication card may look like this on one side, $\begin{array}{r}7\\ \underline{5}\end{array}$ and like this on the other $\begin{array}{r}7\\ \underline{5}\\ 35\end{array}$. Likewise, a

division card will be printed on one side in this form, $7\overline{)35}$, and on the reverse side in this manner, $\begin{array}{r} 5 \\ 7\overline{)35} \end{array}$. The 81 multiplication cards will be indistinguishable from the 81 addition cards on the side without answers.

It may be added that in using flash cards for class drill, one ordinarily holds the pack in the left hand where the class can easily see and, with the right hand, takes the cards, one at a time, from the back of the pack and places them at the front. Since the combinations are printed on the back of the card as well as the front, the teacher can see the combination flashed without looking around at the other side of the card, or turning the card. Some teachers begin with the cards upside down and roll or slide them over the top of the pack, from the back to the front, thus inverting each card as it is turned and letting the pupil see it in the proper position. This method is particularly good if one is using cards which are printed on one side only. The teacher must, of course, read the combinations upside down; she needs to be careful not to confuse 9's and 6's.

Useful games and devices. A great many games are available for drilling on the facts of multiplication and division. Lack of space prohibits the description of a large number here, but a few, selected from a variety of sources, will be given.

Hunting Wild Geese

Arrange the nine digits on the blackboard in an angle representing a number of wild geese in characteristic flight formation.

6
2
5
8
4
3
9
1
7

Let the digits occur in random order, and let the order be changed occasionally. As a bullet, use any number from 1 to 9. If the bullet is 7, the pupil wings the geese by naming, in rapid succession, the products, 42, 14, 35, 56, 28, 21, 63, 7, 49. The order in which the products are named may be changed at will; the pupil may begin with 49 and end with 42.

Hunting Wild Geese provides an interesting means of getting rapid vigorous drills on the multiplication combinations. Many pupils can take part, for the time required for one pupil to shoot his geese is very brief. This game can be used when only a portion of the combinations have been presented if care is taken to limit it to those which have been taught. To make the flock of geese larger, some of the digits may be repeated.

As the pupils recall the products more and more easily greater emphasis may be placed upon the time required to give the products. Pupils may vie with each other to see who can give the products in the least number of seconds. They may also be encouraged to strive to lower their own records.

Hunting Wild Geese can also be used for division combination drill. Let the dividends be the geese while the divisor serves as a bullet. The flock of geese will appear as shown. In the later

```
            42
          14
        35
      56
    28
      21
        63
          7
            49
```

work the same scheme may be used for practice on expressing quotients and remainders. That is, the flock of geese may appear in the following form and the pupil, dividing by 7, will say, "5 and 2 left, 3 and 5 left," etc.

37
26
19
51
60
34
25
43
18

Telling the Whole Story

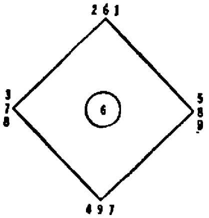
The teacher says, "Tell me the whole story of 4 and 6" and the pupil responds, "4 times 6 is 24; 6 times 4 is 24; 4 into 24, 6 times; 6 into 24, 4 times" or in whatever form he has been trained to give products and quotients. This device may be used for written drills, the teacher dictating the combinations and the pupils writing the four forms $\begin{array}{r}6\\ \underline{4}\\ 24\end{array}$, $\begin{array}{r}4\\ \underline{6}\\ 24\end{array}$, $\begin{array}{r}6\\ 4\overline{)24}\end{array}$, and $\begin{array}{r}4\\ 6\overline{)24}\end{array}$. After a number of combinations have been given, they may exchange papers and check each other's work.

Naming Factors

The teacher chooses a multiple, as 24, and asks, "What makes 24?" The pupils respond, orally or in writing, "4 times 6 is 24; 6 times 4 is 24; 8 times 3 is 24; 3 times 8 is 24."

A Home-Run

Draw on the blackboard a figure representing a base-ball diamond. Write three numbers at each corner, or base, allowing some of the larger numbers to be repeated. Write the multiplier in the pitcher's box. The pupil makes a home run by naming the products of the number in the pitcher's box by the numbers at the bases to which the teacher points. For example, the teacher, starting at the home plate, may point to the numbers 2, 7, 4, 9; the pupil responds 12, 42, 24, 54. If he



makes a mistake, he is called out at that base, as "out at third." Scores may be kept for individuals or for teams which have been chosen.

Supplying Missing Numbers

This drill device affords variety in stating the various facts of multiplication and division. It gives practice in the equation form of statement and in the reading of signs. Mimeographed sheets of exercises can be used for written drill, or, if necessary, examples can be written on the board for the pupils to copy, putting in the missing numbers as they copy the examples, or to give orally. Teams of pupils may be chosen, the one setting examples for the other. Some of the possibilities are:

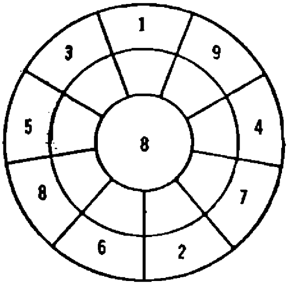
$$\begin{array}{l}
4 \times 9 = \text{—} \\
36 \div \text{—} = 4 \\
36 = \text{—}\ 4\text{'s} \\
\text{—} \times 4 = 36 \\
\text{—} \div 9 = 4 \\
\text{—} = \text{nine } 4\text{'s} \\
9 \times \text{—} = 36 \\
36 \div 4 = \text{—} \\
4 \times \text{—} = 36 \\
36 \div \text{—} = 9 \\
\text{—} = \text{four } 9\text{'s} \\
36 \div 9 = \text{—} \\
\text{—} \div 4 = 9 \\
9 \times 4 = \text{—} \\
\text{—} \times 9 = 36
\end{array}$$

We have given 15 statements for the same teaching unit and have not exhausted the possibilities. Similar lists may be made for the

remaining 44 teaching units. Rather than have a list of statements pertaining to a single teaching unit, it is better to have statements from several teaching units mixed in one list.

Number Wheel

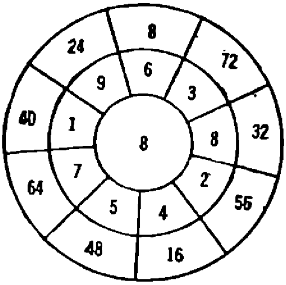
Draw a number wheel on the board, as shown. The nine digits are written in random order in the outer ring and the multiplier in the center. Both the outer digits and the multiplier can easily be changed. The teacher (or another pupil) points to the digits in order, going around the wheel in either direction, and the pupil gives the products. To use this device for division, put the products in the outer ring to serve as dividends.



This device is often called "Race Track"; the pupils are urged to race around the circular track.

As an interesting variation, the products can be written in the outer ring, the multiplicands in the second ring, and the multiplier in the center. The teacher points to a multiplicand, as 4, and two pupils see who can first point to the product, 32. In using the device in this manner, choose two pupils who are nearly equal in ability. The class may be divided into two teams and contest in pairs until all have participated, scores being kept to determine the winning side.

To drill on the division facts, the teacher will point to a number in the outer ring and the pupil first finding the corresponding number in the second ring (the quotient) will win a point for his team.



There are many forms of the racing game. Some, described in Chapter IV, can be used for practice on the multiplication and division facts as well as on the addition and subtraction facts.

Other games given in Chapter IV can be used for the multiplication and division facts. Many more will be found in the references listed at the end of this chapter. The teacher should now be able to select games so as to provide for a maximum of educative value.

SOLVING PROBLEMS

We have pointed out, in connection with our discussion of the teaching of the addition and subtraction facts, the importance of the frequent use of problems whose solution involves the skills which have been acquired. Primary children like to solve problems if the problems are interesting and if they possess the necessary knowledge and skill. While the multiplication and division facts are being mastered, the teacher should select or formulate problems for each of the teaching units, except the ones. Efforts to frame problems for the ones are likely to result in unreal or fantastic statements of little value for teaching purposes.

The following 16 problems are offered as suggestions of the kind the teacher will wish to use. Eight of these involve multiplication and eight, division. For each of the operations, we have used all of the one-place numbers, except 1, as multiplier

and as divisor. Each of these numbers also occurs as multiplicand and as quotient.

1. John makes 2 cents on each *Saturday Evening Post* he sells. Yesterday he sold 6. How much did he make?
2. At the 9-cent sale, mother bought 8 pans. How much did they cost?
3. William's birthday comes 4 weeks from today. How many days until William's birthday?
4. Mildred's mother gives her 3 cents each time she does the dishes all by herself. Last week, she did them 7 times. How much did she earn?
5. The Johnsons live in the country and keep a cow. They sell the milk in quart bottles. Last evening the cow gave 2 gallons of milk. How many bottles of milk did the Johnsons have to sell?
6. Betty's mother bought a crate of strawberries. When the crate was opened, Betty said: "There are 8 baskets of strawberries in each layer and 3 layers. I can tell how many baskets there are in the crate." Can you?
7. Kenneth's father paid him 5 cents each Saturday for carrying out the ashes. Kenneth saved the money for Christmas. When Christmas came, he had carried the ashes out 9 times. How much money had he earned?
8. Margaret took 5 Christmas packages to the post office to mail them. Each one took 6 cents postage. How much did she have to pay?
9. Richard asked his father how much gasoline their truck used. His father said, "I used 5 gallons to go 45 miles." How far did they go on one gallon?
10. Tom, Jack, Harry, Billy, Dick, and Frank helped Mr. Jones clean up his orchard. When they were through, Mr. Jones said, "Boys, here is a basket of apples that you can divide among you." They counted the apples and found that there were 42. How many did each boy get?

11. Ruth wrote two long letters. When she mailed them she had to pay 8 cents postage. How much postage did each letter take?
12. There were 8 boys and girls at Lucile's birthday party. Her mother baked 24 cookies for the party. How many cookies were there for each one?
13. Harry's father pays him 35 cents a week for taking care of the furnace. How much does Harry earn a day?
14. Mother came home from the 9-cent sale and said, "I spent 72 cents." How many things did she buy?
15. Tom rode his bicycle over to Jack's house in the morning, rode back home at noon, rode over again in the afternoon and home in the evening. His cyclometer then showed that he had ridden 8 miles. How far is it from Tom's house to Jack's house?
16. Mary noticed that her mother dropped 18 cents in the box for 3 street car fares. How much was 1 fare?

TESTS ON THE FUNDAMENTAL FACTS

After a few teaching units have been presented, a test should be arranged including all of the multiplication and division facts so far taught. Tests should be frequent as the pupils progress through the 45 teaching units and the results should be recorded in the manner indicated in Chapter IV. Good teaching demands that the progress of each pupil be carefully checked from time to time and that test results be carefully analyzed; there is no other way to detect all weak points and plan proper remedial instruction.

In some tests, the multiplication facts only will be included. Thus, the following test is designed to cover the multiplication facts of the first nine teaching units. Note that the items are not arranged in the order in which they occur in teaching.

2	4	6	2	8	7	2	2
7	2	2	9	2	2	2	6
—	—	—	—	—	—	—	—

3	2	2	2	3	9	2	5
2	5	4	8	3	2	3	2

Also, some tests will include only division facts. The division facts corresponding to the 16 multiplication facts in the above test are included in the following test.

7/14	2/10	6/12	2/8	2/6	9/18	2/12	3/6
5/10	2/14	3/9	8/16	2/18	4/8	2/16	2/4

But many of the tests which the teacher prepares will include both multiplication and division facts. The following test contains 32 facts derived from a mixed group of nine teaching units. Again the items are arranged in random order.

5/20	4 9	1/8	9/9	6 5	1 1	7/35	5 4
9 1	6/30	1/9	1/1	9 4	1 8	5 5	5 7
5/30	6/18	5 6	1 9	4 5	5/35	8 1	4/36
5/25	8/8	6 3	4/20	9/36	3/18	7 5	3 6

Concluding statement. The bulk of the multiplication and division facts are usually assigned to the third grade. And fourth grade teachers testify that these facts, particularly the multiplication facts, are not very well known by the pupils who come to them.

If the multiplication facts are limited to 9 × 9, if they are taught as a short method for addition, if an abundance of interesting drill is provided, if the more difficult of these are made the object of special attention, and if the two facts derived from a combination are taught together, it seems reasonable to expect third grade teachers to bring their pupils to complete control over these important fundamentals. If, on the other hand, the assignment includes all combinations to 12 × 12, if the tables are memorized in serial order, and if the drill lessons are uninteresting or give but little practice on these bonds, we can expect that the job will be poorly done. We have seen that there are 45 multiplication combinations to be learned. Nine of these are ones, eight are twos, and seven more are fives. These, teachers say, pupils easily learn. We have left, then, just 21 combinations in which the difficulties seem to lie. Expressed in equation form, they are:

3 × 3 =	4 × 6 =	6 × 9 =
3 × 4 =	4 × 7 =	7 × 7 =
3 × 6 =	4 × 8 =	7 × 8 =
3 × 7 =	4 × 9 =	7 × 9 =
3 × 8 =	6 × 6 =	8 × 8 =
3 × 9 =	6 × 7 =	8 × 9 =
4 × 4 =	6 × 8 =	9 × 9 =

Surely, we can expect pupils to master 24 easier combinations and these 21 more difficult combinations before they leave the third grade. To suggest how this may be accomplished has been the aim of this chapter.

EXERCISES

1. What are the advantages of writing the multiplication combinations in the vertical form? In the equation form?

2. Why did we not include the zero combinations in the fundamental combinations of multiplication and division?

3. Summarize the argument for teaching the multiplications to 9 × 9. To 10 × 10. To 12 × 12. Which is the more important for a pupil to know, 5 × 11, or 5 × 15?

4. Why should the multiplication combinations not be learned as tables in serial order?

5. Is there better reason for memorizing multiplication tables than addition tables in serial order?

6. Why did we begin with the twos in multiplication? Why should we not have begun with the ones?

7. There are excellent reasons for beginning with the fives. What are they?

8. According to Clapp's study, 1×1 ranks eleventh out of the 81 multiplication facts in difficulty, while $1 \div 1$ ranks fourth in the 81 division facts. Why should this teaching unit be so difficult?

9. What is the advantage of showing the pupil that multiplication is abbreviated addition?

10. What are the characteristics of a good drill device?

11. Do you believe that the following statement is true? "The current tendency in primary education is too much toward entertaining children and not enough toward educating them." Think of the statement with particular reference to the teaching of arithmetic in the primary grades.

12. Describe a game which you would recommend for drilling on the primary facts of multiplication or division.

13. Make up a problem which you would use in teaching $\begin{array}{r}7\\ \underline{5}\end{array}$ and $\begin{array}{r}5\\ \underline{7}\end{array}$. In teaching $6\overline{)48}$ and $8\overline{)48}$.

14. Prepare a test covering the last group of nine teaching units.

15. Do you believe that a primary supervisor is justified in holding a third grade teacher to "strict accountability" for the 45 teaching units of multiplication and division?

16. In what kinds of situations would you have pupils practise on the ones in multiplication and division?

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CHAPTER VIII

ELEMENTARY WORK IN MULTIPLICATION

In addition to learning the primary facts of multiplication and division, third grade pupils will ordinarily be taught to do multiplication examples and to solve problems involving the multiplication skills which have been acquired. They will also have opportunities to put to practical use their skills in division. It is the purpose of this chapter to discuss the teaching of multiplication so far as instruction in this subject is attempted in grade three. Chapter IX will contain a similar discussion of the teaching of division.

Early use of examples. In the first place, it should be emphasized that pupils will be given practice in the solution of examples and problems in multiplication, *while the facts of the 45 teaching units are being learned*. The best textbooks of the present day have pupils solve scores of multiplication examples in which the multiplicand has two digits and the multiplier, one digit, long before the more difficult combinations are learned. Some give the easier three-digit multiplicands while the combinations are being mastered. To teach the entire 162 multiplication and division facts with no opportunities for application except the simple type of problems suggested in the preceding chapter is almost certain to mean loss of interest and consequent ineffective learning.

At this point it should also be said that elementary work in multiplication and division will go hand in hand. The fact that these topics are treated in separate chapters here should not lead the teacher to believe that they are so completely separated in teaching; they are discussed separately merely for the sake of convenience.

Work limited to one-digit multipliers. We shall assume that the third grade work in multiplication is to be limited to one-digit multipliers. This means that the technique of training pupils to write second and later partial products in the proper positions and of adding partial products to obtain the complete product, will be left for the fourth grade.[1] We shall take up multiplication with one-digit multipliers under three heads: (1) multiplication without carrying; (2) multiplication with carrying; and (3) zero difficulties. The application of each of these types of examples to verbal problems will then be shown.

Multiplication without carrying. Let us consider, first, the case of two-digit multiplicands. Obviously, certain very definite limitations must be placed upon the second multiplicand digit, if carrying is to be avoided. If the multiplier is 1, the second multiplicand digit may, of course, be any number from 1 to 9; if the multiplier is 2, the second multiplicand digit may be 1, 2, 3, or 4; if the multiplier is 3, it may be 1, 2, or 3; if the multiplier is 4, it may be 1, or 2; if the multiplier is 5, 6, 7, 8, or 9, the second multiplicand digit must be 1. (As noted above, zeros in the multiplicand are considered later.)

Assume that a class has been taught the 16 multiplication facts of the first group of nine teaching units. The following 29 examples, containing two-digit multiplicands, can now be used.

22	32	42	52	62	72	82	92	23	33
2	2	2	2	2	2	2	2	2	2
43	53	63	73	83	93	24	34	44	54
2	2	2	2	2	2	2	2	2	2
64	74	84	94	22	32	23	33	22	
2	2	2	2	3	3	3	3	4	

[1] See the author's *Teaching Arithmetic in the Intermediate Grades*. Silver, Burdett and Company, 1927, pp. 65-71.

As the number of combinations taught increases, the number of two-digit multiplicands which can be used becomes rapidly greater. When we have taught the entire 45 multiplication combinations, we can use a total 207 two-digit multiplicands in examples without carrying. That is, we can multiply any two-digit number ending in 1, 2, 3, 4, 5, 6, 7, 8, or 9 by 1, making 81 examples; we can multiply any two-digit number ending in 1, 2, 3, or 4 by 2, making 36 examples; we can multiply any two-digit number ending in 1, 2, or 3, by 3, giving us 27 examples; when the multiplier is 4, we have 18 examples; for the multipliers, 5, 6, 7, 8, and 9, we have 9 examples each. This makes a total of 207 examples.

To use the entire 207 examples means considerable overlearning on the ones. It will be best, then, for the teacher to select from these 207 examples those which will distribute the practice fairly well over the 81 multiplication facts. There will be some overlearning of the easier facts because of their frequent occurrence in the higher decades to avoid carrying and considerable emphasis upon the harder facts later to make sure that they are well learned, but, on the whole, the practice can be so distributed over the 207 examples that no bond will be neglected and that the bonds which are more difficult to establish will be given sufficient exercise.

The first set of practice examples in multiplication has been designed to give practice on each of the 81 multiplication facts. To do this in examples having two-digit multiplicands, without carrying, requires a minimum of 63 examples. In these, there is much overlearning of the ones, for when the multiplier is 5, 6, 7, 8, or 9, the second digit of the multiplicand must always be 1. A much more desirable distribution of practice on these 81 facts can be provided after carrying in multiplication has been taught.

A few three-digit multiplicands may also be used, before multiplication with carrying is taught. Examples like $\begin{array}{r}243\\ \underline{2}\end{array}$ or $\begin{array}{r}632\\ \underline{3}\end{array}$

PRACTICE EXAMPLES IN MULTIPLICATION. SET I

The 81 Facts, No Carrying

21	81	42	91	41	81	63	61	51
5	7	4	8	6	9	3	7	5
74	51	72	61	81	46	51	62	41
2	8	3	9	6	1	7	4	5
83	71	61	71	82	71	41	81	41
2	8	6	5	4	9	7	8	9
75	92	71	61	71	31	52	61	13
1	2	4	8	7	9	4	2	1
81	41	52	91	91	31	21	98	82
5	8	2	7	3	7	8	1	3
31	91	91	42	51	22	51	91	31
6	5	9	3	9	1	6	4	5
71	31	32	91	21	21	21	52	61
6	8	4	6	9	7	6	3	5

may help in enlarging the pupil's understanding of multiplication while the combinations are being learned.

The teacher should not forget the importance of introducing examples through the medium of real and interesting problems. Rather than assign $\begin{array}{r}33\\ \underline{2}\end{array}$ as a multiplication example, independent of any concrete origin, it is much better to recite the case of the boy who made 2 cents on each *Saturday Evening Post* he sold and suggest that we find how much he made by selling 33 copies. Pages of examples for drill may quite properly be used provided

that the first examples of the kind have been derived from real problems and that the pupil has been supplied with a motive for learning to solve such examples.

Multiplication with carrying. Teaching children to carry in multiplication is quite like teaching them to carry in addition. Since carrying in addition has been taught and they have, presumably, become proficient in it, there should be no serious difficulty in getting children to see the meaning of, and reasons for, carrying in multiplication.

Let us begin with a problem. "For her birthday party, Eleanor's mother let her take 7 of her little friends to the movies. Eleanor bought 8 tickets, one for herself and 7 for her friends. Each ticket cost 15 cents. How much did she have to pay for the 8 tickets?" After reading the problem or writing it on the blackboard, the lesson may proceed as follows.

Teacher: "If 1 ticket costs 15 cents, how shall we find the cost of 8 tickets?"
Pupil: "Multiply 15 by 8."
Teacher: "Then we set the example down like this. Now think of the 15 cents as 1 dime and 5 cents. How many cents is 8 times 5 cents?"

```
15 cents
 8
——
```

Pupil: "40 cents."
Teacher: "But 40 cents is equal to how many dimes?"
Pupil: "4 dimes."
Teacher: "Then we shall write down 0 for cents and carry the 4 dimes to the dimes' column, just as we carried in addition. Now, how many dimes is 8 times 1 dime?"
Pupil: "8 dimes."
Teacher: "And the 4 dimes that we had to carry makes how many dimes?"
Pupil: "12 dimes."
Teacher: "Then our answer is 120 cents, or $1.20."

Proceeding at once to another problem, the teacher may say: "If I bought 8 gallons of gasoline for my car and paid 23 cents a gallon, how much did it cost? This is the way to find out.

Think, 'eight 3's are 24.' Write 4 and carry 2.
Think, 'eight 2's are 16.' Then add 2 to 16 and write 18.
So our answer is 184 cents, or $1.84.

```
  23 cents
   8
 ——
$1.84
```

"Remember, then, if the first number we multiply gives us more than 9, we write the second figure and carry the first. We add this carry number in after we multiply the next number."

The class will then proceed to the solution of several examples involving carrying. The work will be limited to two-digit multiplicands until the pupils can do such examples accurately and easily.

With a careful initial presentation and sufficient practice pupils should soon become proficient in multiplication with carrying. There is one type of error which the teacher needs to watch for, however. In the example on the right, the pupil did his multiplying correctly, but added in the carry number *before* performing the second multiplication. This is a most natural error for a pupil to make, for in addition he has been taught to add in the carry number before proceeding to the addition of the next column. It is not recommended that teachers should caution pupils about this error for many of them will never think of it unless it is called to their attention; but teachers should be ready to recognize this error if it occurs and give the proper instruction.

$$\begin{array}{r} 43 \\ \underline{6} \\ 308 \end{array}$$

The pupils are now ready to solve examples and problems in which the multiplicand has 2, 3, or 4 digits and the multiplier, 1 digit, with and without carrying. The pupils should be cautioned to the effect that sometimes they will carry and sometimes not. They will then be given examples like $\begin{array}{r} 423 \\ \underline{4} \end{array}$ as well as examples like $\begin{array}{r} 453 \\ \underline{4} \end{array}$. In the former example, the pupil carries but once; in the latter example, twice.

In supplying the pupils with examples for practice, we may at first provide that the primary facts so far taught are used equally often. Later, the easier combinations may occur less frequently and the drill may be directed more specifically at the recently taught, more difficult combinations. The sets of practice examples which follow illustrate how sets of examples may be pre-

pared as the combinations are mastered. It is assumed that the pupils have been taught carrying.

In Set II each of the 49 facts included in the first three groups of teaching units (27 teaching units) occurs once and the combinations $\begin{array}{c}1\\ \underline{1}\end{array}$, $\begin{array}{c}6\\ \underline{5}\end{array}$, $\begin{array}{c}9\\ \underline{2}\end{array}$, $\begin{array}{c}9\\ \underline{4}\end{array}$, and $\begin{array}{c}5\\ \underline{7}\end{array}$, twice. This set will make an excellent review for use when these 27 teaching units have been presented.

PRACTICE EXAMPLES IN MULTIPLICATION. SET II

The Facts of 27 Teaching Units

21	13	26	54	45	87	15	24	56
6	5	3	2	1	2	7	9	5
21	72	31	74	35	29	52	53	31
8	1	1	5	3	2	4	6	4
63	19	94	14	68	26	19	14	25
2	1	4	3	1	5	2	9	7

In the third set, only one of the two facts for each of the ones and twos is practised. This set includes the first 36 teaching units—all but the last group of nine. The two facts for each of the combinations except the ones, the twos, and the doubles are included, two of those in the last group of nine teaching units occurring twice.

Set IV is designed to be used after the entire 45 teaching units have been presented. The set provides practice for 66 of the 81 facts, there being but one fact included for each of those combinations in which 1 or 2 occurs. The facts for the ones and the twos are the reverse of those used in Set III; for example, we used $\begin{array}{c}1\\ \underline{6}\end{array}$ in Set III and we use $\begin{array}{c}6\\ \underline{1}\end{array}$ in Set IV. All of the facts of the last group of nine teaching units occur twice in this set.

PRACTICE EXAMPLES IN MULTIPLICATION. SET III

The Facts of 36 Teaching Units

37 5	42 8	51 6	13 4	64 7	35 9	45 2	81 1	13 3
15 7	23 6	69 4	86 5	97 3	47 6	78 4	54 5	25 1
64 3	32 7	41 9	32 2	45 4	58 3	53 8	98 5	32 9

PRACTICE EXAMPLES IN MULTIPLICATION. SET IV

Summary of Facts

86 2	48 7	54 6	68 8	64 3	83 9	98 5	54 4	19 1	93 7
78 6	19 5	15 8	97 9	89 3	32 4	79 8	97 2	23 3	68 4
96 6	36 1	45 5	67 3	79 4	46 9	12 2	23 5	43 8	76 5
74 1	65 7	89 9	98 6	79 7	67 8	87 7	98 8	76 9	65 3

Set V is designed for intensive practice on all the combinations except the ones, twos, and fives—a group of 36 facts. Each of these 36 facts occurs twice in the set.

Set VI contains three-digit multiplicands. This set, like Set V, is intended to give intensive practice on the 36 facts which remain after the ones, twos and fives have been eliminated. Each of these 36 facts occurs twice. The set also contains, however,

PRACTICE EXAMPLES IN MULTIPLICATION. SET V

The 36 Harder Facts

84 8	68 6	48 3	97 9	79 7	74 4	86 7	79 6	48 9
69 8	93 3	98 4	34 7	73 8	36 9	43 6	63 4	76 3
43 3	39 8	93 7	67 6	74 7	97 3	98 6	73 4	43 6
48 8	73 9	86 3	96 9	49 4	86 4	68 7	48 9	76 8

18 of the 24 combinations involving the ones, twos, and fives—one fact for each of these 18 combinations.

PRACTICE EXAMPLES IN MULTIPLICATION. SET VI

Three-Digit Multiplicands

963 4	849 6	389 7	693 9	687 3	674 8
367 6	419 1	934 3	389 8	478 4	647 7
374 6	512 8	843 8	846 7	215 3	368 9
973 7	698 6	697 8	384 4	479 9	486 3
215 5	679 4	487 9	749 2	397 3	512 6

The teacher can prepare many more sets of practice examples similar to these. A careful record should be kept of the facts included in order that the amount of practice given to each of these fundamental bonds may be known. Only in this way can we be sure that no one of these bonds is allowed to become atrophied through disuse.

The zero difficulties. If the third grade multiplication work is limited to one-digit multipliers, zeros can occur in the multiplicand only. Now, had we included the zero facts with the other multiplication facts, we should have had a total of 100 facts to learn instead of 81. But 10 of the 19 zero facts involve zero in the multiplier. These cannot be used until multipliers of two or more digits are introduced. Since we are not using two-digit multipliers in the third grade, we have here an added argument for postponing the teaching of the zero facts in multiplication until the remaining 81 facts have been learned and used in multiplication examples.

Indeed, we treat the case of multiplying a number by zero quite differently from multiplying zero by a number. A pupil will actually have to think, "eight times zero" in the example, $\begin{array}{r}406\\ \underline{\quad 8}\end{array}$, but when zeros occur in the multiplier, as in this example, $\begin{array}{r}261\\ \underline{\ 20}\end{array}$, he does not have to think "zero times one, six, and two" but writes 0 at once as the last digit of the product and proceeds to multiply by 2. Likewise, in the example, $\begin{array}{r}534\\ \underline{105}\end{array}$, he will react to zero by setting down the second partial product two steps to the left of the first partial product instead of one.

It seems clear, then, that 10 of the 19 zero facts do not have to be taught as facts at all and that the remaining 9, those in which zero is multiplied by a number, can best be taught through the use of examples in which they occur and by teaching the general principle that *zero multiplied by any number equals zero.*

The meaning of multiplying zero by a number can be made

clear by reference to addition. In addition, we learned that *any number plus zero equals that number* and this included the special case that *zero plus zero equals zero,* or, that any number of zeros added together equals zero. Referring to a concrete situation, the lesson may proceed in the following manner.

Teacher: "Dick has 4 pockets in his sweater. In the first pocket there are no apples, in the second pocket there are no apples, in the third pocket there are no apples, and in the fourth pocket there are no apples. How many apples has Dick in his sweater pockets?"

Pupil: "Not any."

Teacher: "We can say it this way. There are zero apples in the first pocket, zero apples in the second pocket, zero apples in the third pocket, and zero apples in the fourth pocket. If we add 4 zeros, what do we get?"

$$\begin{array}{r} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \end{array}$$

Pupil: "We get zero."

Teacher: "Then 4 zeros are how many?"

Pupil: "4 zeros are zero."

Teacher: "If Dick had had five pockets and zero apples in each pocket, how many apples would he have had all together?"

Pupil: "Zero."

Teacher: "Then 5 zeros are how many?"

Pupil: "5 zeros are zero."

Teacher: "And how many are 6 zeros?"

Pupil: "6 zeros are zero."

In this manner, the pupils see again that any number of zeros added together gives zero for a sum and they discover that zero multiplied by any number gives zero for a product.

Teacher: "Now let us try an example in which we must multiply zero by a number. Mrs. Johnson bought 2 sacks of flour at $1.60 each. How much did they cost? We write the example this way.

$$\begin{array}{r} \$1.60 \\ 2 \\ \hline \$3.20 \end{array}$$

2 zeros are how many?"

Pupil: "2 zeros are zero."

Teacher: "We write zero under the 2. Now, 2 sixes are how many?"

Pupil: "2 sixes are 12."

Teacher: "What do I write?"

Pupil: "Write the 2 and carry the 1."

Teacher: "2 ones are 2 and 1 makes 3. Then how much did the 2 sacks of flour cost?"

Pupil: "They cost $3.20."

Several additional examples, illustrating the multiplication of zero by a number, will be worked on the blackboard. These examples will have various multipliers and zeros in various places in the multiplicand. Then the pupils will be given examples to work for themselves and the teacher will closely supervise their early work to make sure that wrong procedures are not initiated.

Immediately after they first encounter zeros in multiplication, the pupils should be given intensive practice on the various types of examples in which this difficulty appears. When this difficulty has been well mastered, practice may be less intensive and occur at greater intervals.

The following set of practice examples is typical of the kind that may be used. In this set, each of the nine zero combinations occurs twice and five of them occur three times. The multiplicands are so selected that no other combination occurs more than once—that is, the set does not include both facts for any combination. Each digit, except 1, occurs twice as a multiplier; 9 occurs three times.

PRACTICE EXAMPLES IN MULTIPLICATION. SET VII

The Zero Facts

420 5	600 3	704 8	410 6	730 9	209 4
400 7	380 2	900 1	806 5	960 8	206 9
300 4	760 6	590 9	205 7	600 2	501 3

Solving problems. The teacher will be continually on the alert for good problems. These problems will be drawn from a wide variety of sources; they will be descriptive of many kinds

of situations with which children are acquainted. An effort should be made to find or invent problems that involve the various multiplication skills which have been discussed in this chapter. The following problems are typical and are offered as suggestions.

1. Mr. Jones bought 2 bags of fertilizer for his garden. Each bag weighed 150 pounds. How much did the 2 bags weigh?
2. John gets $6.00 per month for taking care of a neighbor lady's furnace. How much does he earn in 5 months?
3. At a sale, Mr. Dixon bought 4 tubes for his radio at $2.48 each. What did the four tubes cost?
4. What does Mrs. Brown have to pay for 3 pounds of steak at 29 cents a pound?
5. The fare to the city is $1.05. What would be the fare for a party of 8 people?
6. Robert delivers 45 papers each evening, except Sunday. How many does he deliver in a week?
7. A group of 9 boys formed a baseball team. What did the suits cost at $2.75 each?
8. Mildred counted up on the calendar the number of weeks until her birthday. She found that it was just 18 weeks. How many days until Mildred's birthday?

Checking results. In their early work in multiplication, pupils will be taught to check results. Neither the answers to problems nor those to examples should be accepted as satisfactory until the pupil has assured himself by proper checking that those answers are correct.

The best check for pupils in the primary grades to use for multiplication seems to be the reverse of that process, or division. This check cannot be used for early work with one-digit multipliers, however, unless the elementary work in multiplication and that in division are presented simultaneously. We have indicated in Chapter VII that we would teach the multiplication

and division facts together and, in the third paragraph of the present chapter, have stated that "elementary work in multiplication and division will go hand in hand." From the very beginning of his work in multiplication, then, the pupil will check his results by dividing the product by the multiplier and comparing the resulting quotient with the multiplicand.

It is not necessary, however, that all checking of multiplication be done in this manner. The pupil may also check his answer by repeating, carefully, the steps of his work in multiplication. In repeating his work, he may be taught to multiply in the direction opposite to that first taken. If he, on the first multiplication, thought "seven 6's are 42," he will now think, "six 7's are 42," and "three 7's are 21 and 4 are 25," rather than "seven 3's are 21 and 4 are 25," etc. Thus, when he repeats his multiplication, he practises a different set of facts—the same combinations in the reverse form. Pupils should be trained in the use of both these methods of checking examples in multiplication.

$$\begin{array}{r} 436 \\ \underline{7} \\ 3052 \end{array}$$

Which way should we multiply? At this point, it may be well to consider, briefly, the two ways of multiplying which were mentioned in the preceding paragraph. Should we multiply up or should we multiply down? That is, in the example shown, should pupils be trained to think, "six 4's are 24" or "four 6's are 24"?

$$\begin{array}{r} 384 \\ \underline{6} \\ 2304 \end{array}$$

At first thought, the question seems somewhat like that of adding up or down. We have given, in Chapter V, our reasons for preferring that pupils be taught to add downward. These reasons apply to multiplication, but with somewhat less force, for, in addition, we are dealing with columns of varying heights, whereas, in multiplication, we have but two numbers, in addition to the answer. However, when first learning multiplication, the pupil should be taught to multiply up, for it is the multiplicand which is multiplied, not the multiplier. Furthermore, the zero difficulties are different in multiplying down. After the pupil has fully learned the meaning of multiplication, it matters little

in which direction he multiplies but he will probably continue to multiply in the direction he has been taught, namely, upward. It may be well to teach pupils to multiply in both directions, eventually, in order that one may be used as a check upon the other, as was pointed out in a preceding paragraph.

EXERCISES

1. How soon would you use examples having two-digit multiplicands and one-digit multipliers, no carrying?

2. What limitations are placed on the multiplicand digits, if there is to be no carrying in the example?

3. Describe the method which you would use in teaching the meaning of carrying in multiplication.

4. What error was made in this example? How would you correct the pupil's tendency to make this error?

24
3
92

5. How can the teacher make sure that no fundamental multiplication bond is allowed to become atrophied?

6. Why do "we treat the case of multiplying a number by zero quite differently from multiplying zero by a number?"

7. How would you show that zero times any number is zero?

8. Why should problems, as well as examples, be given early in the pupil's work in multiplication?

9. Write four problems for use in multiplication in grade three. Provide as much variety as possible in both the multiplicands and the multipliers.

10. Why should pupils be trained to check all solutions? How should multiplication be checked?

11. Should we multiply up or down? Why?

12. State and illustrate the difficulty steps which have been recognized so far in the teaching of multiplication. What difficulties must be mastered in the work of later grades?

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Chapter IX

ELEMENTARY WORK IN DIVISION

The curriculum of the primary grades—grades one, two, and three—will not provide for a great amount of work in division. This subject, one of the most difficult which the pupils of the elementary school encounter, usually occupies a prominent place in the course of study of the fourth grade and is given further attention in grades five and six. Ordinarily, however, some instruction in division will be given in the third grade. It is the purpose of this chapter to discuss the teaching of division to the extent that instruction in this subject is undertaken in the third grade.

As a matter of economy in learning, we have provided that the primary facts of multiplication and division should be taught together. As a matter of further economy, the first work in multiplication and division, in which the primary facts find their application, will be given in close sequence. This does not mean that the first multiplication example containing a two-digit multiplicand and a one-digit multiplier will be followed immediately by the corresponding example in division; when a new process is presented, there should be a brief period of intensive practice that the bonds involved may be fairly well fixed. But within a few lessons, division examples of the corresponding degree of difficulty will be introduced, and, thereafter, practice on the two operations will go hand in hand, the one process being used to check results in the other. Thus, multiplication without carrying will be followed by division without carrying; multiplication with carrying will be followed by division with carrying; two-digit multipliers will be followed by two-digit divisors; etc.

The fractional form in division. To divide a number by 2 is the same as to take $\frac{1}{2}$ of that number. This is a fundamental idea in division and should be presented early, in connection with the learning of the primary facts. Previous to this, the pupils must, of course, be made acquainted with the meaning of fraction and with the form in which a fraction is written. Instruction need not go beyond fractions having 1 as a numerator. Acquaintance with fractions will begin in the first grade, where, through concrete means, the meaning of one-half (but not the form of expression, $\frac{1}{2}$) will be made clear, and will grow slowly and become gradually more complete through the succeeding grades. In the third grade, then, when the pupils are learning the primary facts of multiplication and division, they will be ready to use the fraction form as a varied form of statement. For example, having learned that three 4's are 12, that four 3's are 12, that 12 divided by 4 equals 3, and that 12 divided by 3 equals 4, they will also learn that $\frac{1}{4}$ of 12 is 3 and that $\frac{1}{3}$ of 12 is 4. In connection with the learning of each teaching unit, the division facts will be presented in these two forms.

Early use of examples. In the preceding chapter, we suggested that examples and problems in multiplication should be solved while the facts of the 45 teaching units are being learned. Likewise, examples and problems in division should be given while these primary facts are being mastered. This could, of course, be inferred from our previous statements concerning teaching the processes of multiplication and division together.

Work limited to one-digit divisors. It is recommended that the third grade undertake no division examples having divisors of two or more digits. It is very much more difficult to estimate quotient figures when the divisor has two digits than when it has but one. This, the major difficulty in division, will be left for the fourth grade.

Long division and short division. Probably the great majority of schools in this country teach short division before they teach long division. It is customary, in other words, to insist

that all examples having one-digit divisors be done in short division form. The long division form is then taught when two-digit divisors are introduced.

This plan seems to be based upon a poor psychology. Short division is, as its name implies, a short method. It is intrinsically more difficult than long division (for any given example) because in short division one must deal with quantities which he cannot see, while in long division all of the quantities used in the solution are in plain sight. In an example like $4/\overline{84}$, where the divisor has but one digit and the dividend is so constructed that carrying is not necessary, it may be just as easy to do the example by short division as by long, but in most division examples carrying is necessary and the example is, therefore, much more difficult if undertaken by short division. If a pupil is learning to solve the example $\overset{146}{6/\overline{876}}$ by short division, he must think of a 2 left over after the first division, must visualize the 2 which he cannot see prefixed to the 7 which he can see, must think, "How many 6's in 27?", must subtract an invisible 24 from a partially visible 27, etc. This is difficult, very difficult for third grade children. But if the solution is written out in the customary long division form, there is less left to mental imagery, the steps in the process are clearly portrayed, and the pupil will master the elements of a difficult process without the unfortunate condition of initial discouragement. Let us first use the long division form as an easier method of acquiring facility in a difficult process and, later, say in the fourth grade, introduce short division as a short method, as a time saver.

$$\begin{array}{r} 146 \\ 6/\overline{876} \\ \underline{6} \\ 27 \\ \underline{24} \\ 36 \\ \underline{36} \end{array}$$

"But," rejoins the defender of present practices, "if you use the long division form for examples having one-digit divisors, pupils will become so habituated in the long process that they will never use the short division form unless compelled to do so by the teacher." Perhaps not. But, in schools teaching short

division first, the writer has frequently observed upper grade pupils use the long process in dividing by 2. Furthermore, if the choice is between going through life using the long division form for one-digit divisors and learning division well in the third and fourth grades, without the retardation that comes from initial discouragement, the writer will choose the latter. There are many worse arithmetic practices than using the long form in dividing by one-digit numbers.

Attack difficulties one at a time. The best single piece of advice which one can give a third grade teacher with respect to the teaching of division is, *Isolate the separate difficulties and present a graded series of lessons which will introduce these difficulties one at a time.* Remembering that we are confining our discussion here to examples having one-digit divisors, and that we shall have no examples in which the quotient has more than three digits, we may recognize the following classes of examples:

1. The primary division facts, as 8/48.
2. Examples having two-digit and three-digit quotients, no carrying.
   (a) Divisor contained in first digit of dividend, as 3/69, 2/486.
   (b) Divisor contained in first two digits of dividend, as 3/126, 4/2484.
3. The primary facts, with remainders, as 2/17, 6/53.
4. Examples involving carrying, no remainders, as 4/172, 5/2325.
5. Examples involving carrying, with remainders, as 6/232, 7/4538.
6. Zeros in quotient, with and without remainders.
   (a) At end of quotient, as 4/40, 5/750, 3/32, 8/2643.
   (b) In midst of quotient, as 6/1206, 4/1612, 9/5419, 8/3218.

Division without carrying. While the 162 primary facts of multiplication and division are being learned, pupils will frequently be given practice with examples having two- and three-digit quotients, but in which there is no carrying, and which dc not have remainders. The purpose of this work is to give practice on the primary facts of division and to let the pupils become

accustomed to the form of division. The work will be introduced by a problem, such as, "John, Dick and Robert collected old magazines and sold them for 93 cents. How much did each boy get?"

Since there is to be no carrying in these first division examamples, the same limitations are placed on the dividends that were imposed upon the multiplicands in multiplication without carrying. Where the quotient is a two-digit number, the second digit of the quotient must be 1, if the divisor is 9, 8, 7, 6, or 5; it must be 1 or 2 if the divisor is 4; it must be 1, 2, or 3, if the divisor is 3; it must be 1, 2, 3, or 4, if the divisor is 2; it may, of course, be any digit if the divisor is 1. Where the quotient is a three-digit number, these statements apply to both the second and the third quotient figures.

The first set of practice examples is designed to give practice on all the primary division facts, except those in which 1 is the divisor. We omit 1 as a divisor in these examples because dividing by 1 would be meaningless to pupils in the third grade. (Strictly speaking, dividing by 1 is not division at all.) This set provides for an excessive amount of practice on those facts in which the quotient is 1 for the reason stated in the preceding paragraph. When division with carrying has been learned, we can equalize better the distribution of practice on these fundamental bonds.

Set I contains 58 examples. This is the minimum number of examples having two-digit quotients in which practice can be given on each of these 72 primary division facts (all except the nine in which 1 is the divisor).

The primary facts with remainders. As a preparatory step to division with carrying, pupils should be given much practice with examples having one-digit divisors and one-digit quotients with remainders. Problems of the "How-many-for-each-and-how-many-left-over?" type will be employed frequently. The following problem may be used to introduce pupils to examples of this kind.

PRACTICE EXAMPLES IN DIVISION. SET I

The Primary Facts, Without Carrying

7/357	6/486	9/549	3/216	8/408	2/148
4/248	5/205	7/427	5/255	3/189	9/729
6/246	8/728	4/168	7/567	5/105	3/156
5/305	6/126	7/147	9/189	6/546	4/128
8/248	6/426	5/155	4/364	6/306	9/459
3/126	9/819	5/455	6/186	3/246	8/168
7/217	3/273	7/637	2/104	8/328	5/405
2/122	4/208	9/279	7/497	8/488	4/284
2/184	9/369	8/648	7/287	9/639	4/328
5/355	6/366	8/568	2/166		

When the 4 children came home from school they were very hungry. "You may have the rest of those cookies in the jar," said their mother. The children ran to the jar and found that there were 11 cookies. How many were there for each, and how many left over?

The ordinary arithmetic textbook distributes the practice given to the primary facts with remainders very unevenly. Furthermore, immensely more practice is usually given to the primary facts without remainders than to the primary facts with remainders. Thorndike found that a certain well-known book made provision for dividing 24 by 3, 91 times; 24 by 4, 76 times; 24 by 5, 18 times; 24 by 6, 50 times; 24 by 7, 5 times; 24 by 8, 61 times; and 24 by 9, 1 time. Four of these seven divisions have no remainders; they occur 278 times, an average of 70 times for each. But the three that have remainders occur a total of 24 times, an average of 8. Note, too, that 24 divided

by 5 occurs 18 times, but that 24 divided by 9 occurs but once.[1]

Not only do textbooks give much more practice on divisions without remainders than on divisions with remainders, and on some uneven divisions than on others, but they also tend to give much more practice on the smaller dividends of the primary facts than on the larger. For example, the book referred to in the preceding paragraph[2] made provision for dividing 48 by 5, 7 times; 48 by 6, 17 times; 48 by 7, 4 times; 48 by 8, 33 times; and 48 by 9, 2 times. Here, for the two divisions which come out even, there are 50 practices, an average of 25. The three divisions which do not come out even occur a total of 13 times, an average of 4. These figures should be compared with those given in the preceding paragraph.

There are many of these uneven divisions upon which specific practice should be provided in teaching the primary facts with remainders. We shall divide 2 into 3, 5, 7, 9, 11, 13, 15, 17, and 19; we shall divide 3 into 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26, 28, and 29; etc. Table IV shows the number of examples to be used and the smallest and largest dividends for each divisor. When the divisor is 6, for example, the smallest dividend used will be 7, and the largest, 59. There will be 45 examples for this divisor. These will have, as dividends, all the numbers from 7 to 59, inclusive, except those into which 6 is contained without a remainder. It will be seen that there is a total of 324 examples upon which practice should be given. Practice should be distributed rather evenly over these examples, if the primary facts are equally well known, but if the pupils show more hesitation on those in which the larger numbers occur, additional practice should be provided for those examples.

We could easily give these 324 examples in a series of sets of practice exercises, but they would take up several pages and be of little help to the teacher. The teacher should, however, pre-

[1] Thorndike, Edward Lee. *The New Methods in Arithmetic.* Rand McNally and Company, 1921, p. 72.

[2] *Ibid.*, p. 73.

TABLE IV. *Division Facts with Remainders*

Divisor	Dividends		
	Smallest	Largest	Number
2	3	19	9
3	4	29	18
4	5	39	27
5	6	49	36
6	7	59	45
7	8	69	54
8	9	79	63
9	10	89	72
Total	3	89	324

pare such sets of exercises for herself. To do so, simply write down the 9 numbers into which 2 is to be divided, the 18 numbers into which 3 is to be divided, the 27 numbers into which 4 is to be divided, etc., keeping the numbers for each divisor in separate groups. Then check off the numbers as they are included as dividends in the practice exercises. The various divisors and dividends should be taken in a miscellaneous order. For example, we would not give $6\overline{)13}$, then $6\overline{)14}$, $6\overline{)15}$, etc., in regular order, for the pupil would derive one answer from the preceding, rather than get it by division and subtraction.

Higher decade subtraction. In Chapter VI, on the teaching of subtraction, we did not take up higher decade subtraction because this type of skill is not particularly needed in subtraction itself. But it is needed in short division. Were we planning to teach short division before long division it would now be necessary to stop long enough to teach 175 higher decade subtraction combinations. That is, the pupil would have to learn to subtract 10 from 11, 12, 13, and 14; 12 from 13, 14, 15, 16, and 17; 14 from 15, 16, 17, 18, 19, and 20; 15 from 16, 17, 18, and 19; etc. Pupils who are forced into short division without due provision having been made for higher decade subtraction are likely to develop counting habits in finding differences, and,

because of the subtraction difficulties, will frequently employ the long division form for one-digit divisors, after they have learned it. It seems much better to postpone higher decade subtraction until we are ready to use it in short division and to permit pupils to solve examples in the primary division facts with remainders like this, $\begin{array}{r}8\\7\overline{)58}\end{array}$, rather than insist that they solve them like this, $\begin{array}{r}8\\7\overline{)58}\\56\\\hline 2\end{array}$, and, of course, to have them practise on written examples rather than on oral examples. This procedure seems to be a much better training for the difficult division examples in which they must carry.

To be sure, the 175 higher decade subtraction combinations needed in short division can be learned in connection with higher decade addition. Those in which the minuend is greater than 39 are just 87 in number, and are the exact counterparts of those higher decade addition combinations whose sums are greater than 39 and which are used in carrying in multiplication. They *can* be so learned, but we believe that the plan which we are following will be easier for children to learn by, and that is a prime consideration in teaching a difficult subject.

Two-digit quotients with remainders. After the pupils have become proficient on the primary division facts, with remainders, they should be given examples having two-digit quotients and remainders before carrying is introduced. In such examples, the dividend will contain two digits, like $4\overline{)86}$, or three digits, like $4\overline{)165}$, and $7\overline{)639}$. The last example will, of course, be solved in the manner shown. The pupil thus receives further practice in the form of writing out his solutions, having, in such cases, two steps in the division process. Many examples of this kind can be made but it is not advisable to spend a great amount of time on work of this type because of the difficulty of getting pupils to subtract when carrying is introduced. Pupils who have

$$\begin{array}{r}91\\7\overline{)639}\\63\\\hline 9\\7\\\hline 2\end{array}$$

been overtrained on these examples tend to solve this example as shown, neglecting to subtract the 72 from the 74.

```
  91
8/749
  72
   9
   8
   1
```

Division with carrying. We are planning that the pupil shall move by gradual and comparatively easy steps through the difficulties of division. We do not want him to be overwhelmed and consequently discouraged by tasks which are beyond his ability. If he has had the work which has been outlined in the previous paragraphs, he should now be able to take up division with carrying and master it readily.

Let us again begin with a problem, such as the following.

There are 365 days in a year. How many weeks are there in a year?
Teacher: "This is the way we find out. Think, how many 7's in 36?"
Pupil: "Five."
Teacher: "Write the 5 above the 6 of the 36. Multiply 5 by 7. Write the 35 under the 36. Subtract. Write the 5 after the 1. Think, how many 7's in 15?"
Pupil: "Two."
Teacher: "Write the 2 above the 5. Multiply 2 by 7. Write the 14 under the 15. Subtract. Then, how many weeks are there in 365 days?"
Pupil: "Fifty-two."
Teacher: "And how many days over?"
Pupil: "One."

```
  52
7/365
  35
   15
   14
    1
```

The teacher will, of course, suit her actions to her words. She will write the 5 above the 6 as she says, "write the 5 above the 6 of the 36," etc.

It will be seen that we instruct the pupil to "write the 5 above the 6 of the 36" and to "write the 2 above the 5." There are two reasons for having the quotient digits written in these exact positions: First, it will help the pupil later to keep track of the dividend digits which he has brought down; second, it will help later in locating a decimal point in the quotient correctly. Neither of these reasons would be understood if given now but

we try to get the pupil to form the habit of placing the quotient figures in their proper places since they must get into the habit of placing them somewhere. It is a little like training young children in certain socially approved habits of conduct at the dining table long before they can appreciate the reasons for the practices which we urge.

After other illustrative problems and examples have been solved before the class and the routine, "divide, multiply, subtract, bring down," has become well impressed, the pupils will be given examples to solve. One or two solutions should be left on the blackboard where they can be seen by the pupils as they work. The teacher will move about inspecting the work, checking wrong starts, and offering, where necessary, further advice and explanation.

Some teachers prefer, when introducing division with carrying, to give, at first, examples without remainders and, later, examples with remainders. Since we have already dealt with remainders in examples without carrying, however, remainders should have ceased to be a difficulty. We may, then, in teaching division with carrying, use examples both with and without remainders. The pupils will be warned that sometimes the examples have remainders and sometimes they do not. They will, too, check their solutions, as indicated in a later paragraph.

In this connection, we may remark that most division problems in real life yield remainders. If our purpose in school is to enable pupils to meet successfully the arithmetical situations which arise outside of the school, it would seem to be a mistake to give in school a preponderance of examples without remainders. Many textbooks in arithmetic and many courses of study are so overloaded with examples without remainders that pupils come to look upon a remainder as a sure indication that the example is wrong.

The following facts will help us to get an idea of the proportion of examples which should come out even.

When the divisor is 2, the possible remainders are 0 and 1.

These, in the long run, will occur equally often. But when the remainder is 0, the example comes out even. So, when the divisor is 2, we should expect one-half of our examples to come out even.

When the divisor is 3, the possible remainders are 0, 1, and 2. These, in the long run, will occur equally often. But when the remainder is 0, the example comes out even. So, when the divisor is 3, we should expect one-third of our examples to come out even.

When the divisor is 4, we should expect each of the remainders, 0, 1, 2, and 3, to occur equally often. So, when the divisor is 4, we should expect one-fourth of our examples to come out even.

Likewise, when the divisor is 5, one-fifth of our examples will, in the long run, have no remainders. When it is 6, one-sixth of them will come out even. And so on.

Then, in our early work in division with carrying, if the divisors, 2, 3, 4, 5, 6, 7, 8, and 9, are used equally often, and dividends are selected at random, we can expect less than a fourth of our examples to come out even and more than three-fourths of them to have remainders.[3] The third grade teacher will do well to provide that approximately a fourth of the examples she assigns in division with carrying have no remainders and that examples without remainders shall become less and less frequent as the divisors become larger. In later grades, when pupils are using three-digit divisors, examples without remainders will almost never occur.

There are literally thousands of examples having one-digit divisors and two-digit quotients, with carrying. We can divide 2 into any number from 20 to 199 inclusive, and get a two-digit quotient. This gives us 180 examples. But in one-half of these, there is no carrying (those whose dividends are in the twenties, the forties, the sixties, etc.), so we have, when the divisor is 2, 90 examples with two-digit quotients and in which one must carry. Likewise, when the divisor is 3, we can use any number

[3] Under these conditions, the theoretical frequency of examples without remainders will be 22.86%.

from 30 to 299 as dividend; this gives us 270 examples in 90 of which there is no carrying, or 180 which suit our present purposes. When the divisor is 4, we get 360 examples (using as dividends all numbers from 40 to 399) in 270 of which there is carrying. These facts for all divisors from 2 to 9, inclusive, are shown in Table V. It will be seen that if we wish to give practice on division examples with carrying—examples having two-digit quotients—there are 3,240 different examples which may be used.

TABLE V. *Examples Having One-Digit Divisors and Two-Digit Quotients*

Divisor	Dividends	Number of Examples	
		Total	With Carrying
2	20-199	180	90
3	30-299	270	180
4	40-399	360	270
5	50-499	450	360
6	60-599	540	450
7	70-699	630	540
8	80-799	720	630
9	90-899	810	720
Total		3960	3240

To make systematic provision for practice on each of these 3,240 examples is a task which we can hardly expect either the teacher or the textbook writer to assume. In situations like this, we must expect some transfer of skill. We have reason to believe that such transfer is more likely to occur if the examples which are chosen are selected so as to permit each digit to appear in the first place in the quotient, and, so far as possible, in the second place in the quotient, and remainders to occur in more or less chance order.

To illustrate what we have in mind, two sets of practice examples, containing a total of 72 examples, have been prepared. For the most part, the first set contains examples having the

smaller digits in the first place in the quotient, and the second set, the larger. In the two sets together there are nine examples for each divisor from 2 to 9, inclusive. In each divisor group, the dividends are so chosen that each of the nine digits is used as the first quotient figure, those which can occur in the second quotient place in examples with carrying are used, and that all possible remainders will be found.

PRACTICE EXAMPLES IN DIVISION. SET II

Two-Digit Quotients, with Carrying

3/206	7/325	5/172	2/31	6/224	4/312
8/217	4/139	9/716	3/74	6/334	7/133
9/315	5/284	6/114	2/75	3/238	8/412
7/197	3/171	9/832	4/50	5/391	6/279
4/387	5/473	8/306	2/96	7/389	3/40
7/261	9/415	8/395	5/60	6/169	8/128
2/119	4/225	9/732	2/52		

The teacher can prepare additional sets of practice examples similar to these by simply selecting quotients, divisors, and remainders and multiplying the quotient by the divisor and adding in the remainder to get the dividend. The quotients, divisors, and remainders will be selected so that the nine digits occur equally often in each, so far as this is possible. Of course, the remainder digit can never be as large as the divisor. The second quotient figure cannot be 1, except when the divisor is 6, 7, 8, or 9; it cannot be 1 or 2, if the divisor is 2 or 3; it cannot be 1, 2, 3, or 4, if the divisor is 2.

PRACTICE EXAMPLES IN DIVISION. SET III

Two-Digit Quotients, with Carrying

3/140	4/93	6/389	8/679	9/223	5/116
4/180	6/584	9/125	3/106	5/228	7/574
2/131	7/453	2/196	8/501	6/438	5/335
4/271	2/175	7/641	9/618	4/358	8/760
3/250	5/447	8/590	6/493	3/289	7/517
2/152	9/515				

Turning now to examples having three-digit quotients, we find ten times as many examples as we have with quotients of two digits. Table V shows that there are, in all, 3,960 examples having one-digit divisors and two-digit quotients. For one-digit divisors and three-digit quotients, there are 39,600 examples, as is shown in the following table. This table indicates that when the divisor is 2, we can use as dividend any number from 200 to 1999, inclusive, and have three-digits in the quotient. This gives us 1800 examples, of which 1350 involve carrying. Of the total number of 39,600 examples, 37,620 involve carrying.

These 37,620 examples include those in which zeros occur in the quotient—a type which is discussed in later pages.

We do not expect the primary teacher to be keenly interested in this table. The table is supplied to show the enormity of the task of giving practice on all possible examples having one-digit divisors and three-digit quotients. Only a few of these 37,620 examples will be given to the pupils to solve as a means of gaining facility in dealing with the type of division examples which we are now discussing. However, these few should be selected with care. The impromptu writing of just any examples—a practice which many teachers follow—will not provide adequately

TABLE VI. *Examples Having One-Digit Divisors and Three-Digit Quotients*

Divisor	Dividends	Number of Examples	
		Total	With Carrying
2	200-1999	1800	1350
3	300-2999	2700	2340
4	400-3999	3600	3330
5	500-4999	4500	4320
6	600-5999	5400	5220
7	700-6999	6300	6120
8	800-7999	7200	7020
9	900-8999	8100	7920
Total		39600	37620

for drill on the fundamental bonds involved. Each of the eight digits from 2 to 9, inclusive, should be used as a divisor; each of the nine digits should appear about equally often in the quotient; and the various possible remainders should appear about equally often.

Practice Examples in Division, Set IV, have been prepared to illustrate how examples of this kind should be chosen. The set contains four examples having 2 as the divisor and three examples with each of the remaining one-digit numbers (except 1) as divisor. It was necessary to include four examples with 2 as divisor in order that all nine digits might appear in the quotient, for 1, 2, 3, and 4 can appear in the first place only, if there is to be carrying in each step of the example. Remainders appear in nearly all of these examples, and the same remainder does not appear twice for any one divisor, except 2.

Zeros in the quotient. Zeros in the quotient cause much trouble. In the example, $4\overline{)962}$, pupils tend to write 24 as the quotient, leaving 2 as the remainder. In the example, $7\overline{)4214}$, a very common answer is 62, instead of 602. These errors seem less likely to occur when the examples are solved by long division than when the short division procedure is used, for we can

PRACTICE EXAMPLES IN DIVISION. SET IV

Three-Digit Quotients

3/2810	8/3087	5/933	2/972	6/5591
4/1077	9/2550	2/719	3/741	7/4490
9/6831	6/1724	4/734	3/556	8/1395
6/2740	4/2299	2/357	7/1986	5/2862
7/6705	9/5776	2/592	5/1974	8/2156

rather easily emphasize the point that each time we bring down a number we must write another figure in the quotient. In the former example, we brought down the 2 and must, then write something in the quotient. But since 4 is not contained in 2, we must place a 0 in the quotient. Likewise, in the latter example, when we bring down the 1, we must write something in the quotient, but since 7 is not contained in 1, there is nothing left to do but to place a 0 in the quotient.

```
  240
4/962
  8
  16
  16
   2
```

If this important principle—*every time we bring down a number we must write something in the quotient*—is emphasized when examples containing zero in the quotient are first presented; if there is then a period of intensive practice on examples of this type; and if pupils are required to check all solutions, this zero difficulty will soon be mastered and will cause little trouble in later lessons. But when examples of this category are first presented, there must be a careful explanation of the difficulty, several illustrative examples must be worked out, and there must then be a period of practice which is closely supervised by the teacher.

```
  602
7/4214
  42
    14
    14
```

We shall consider zeros at the end of the quotient and zeros in the midst of the quotient. Zeros at the end of the quotient can

occur with either two- or three-digit quotients; zeros in the midst of the quotient can, of course, occur in three-digit quotients only, remembering that in the third grade we shall not have more than three digits in the quotient.

If the divisor is 2, and the quotient is to have two digits, the second of which is 0, the possible quotients are, of course, 10, 20, 30, 40, 50, 60, 70, 80, and 90. Each of these can occur with or without a remainder of 1. So we have, in all, 18 examples having a 0 in the two-digit quotient when the divisor is 2. They are:

$2\overline{)20}$ $2\overline{)21}$ $2\overline{)40}$ $2\overline{)41}$ $2\overline{)60}$ $2\overline{)61}$ $2\overline{)80}$ $2\overline{)81}$ $2\overline{)100}$

$2\overline{)101}$ $2\overline{)120}$ $2\overline{)121}$ $2\overline{)140}$ $2\overline{)141}$ $2\overline{)160}$ $2\overline{)161}$ $2\overline{)180}$ $2\overline{)181}$

If the divisor is 3, we have the same nine quotients, but each can occur with no remainder, a remainder of 1, or a remainder of 2. This gives us 3 × 9, or 27, examples.

When the divisor is 4, we have the same nine examples, each occurring with no remainder, a remainder of 1, a remainder of 2, or a remainder of 3. When the divisor is 4, then, we have 4 × 9, or 36 examples.

Likewise, when the divisor is 5, we have 45 examples; when the divisor is 6, we have 54 examples; etc.

These facts are summarized in Table VII. The table shows that when the divisor is 2, there are 18 examples having a quotient of two digits, one of which is 0. The total number of such examples is 396.

This table also shows the number of examples having zeros in three-digit quotients. When the zero is at the end of the quotient, as in the example, $\begin{array}{l}\ \ 630\text{–}2\\ 4\overline{)2522\quad}\end{array}$, there are 3960 examples, ten times the number in the preceding column. For zeros in the midst of the quotient, as in the example, $\begin{array}{l}\ \ 705\text{–}3\\ 5\overline{)3528\quad}\end{array}$, there are

TABLE VII. *Zeros in Quotients with One-Digit Divisors*

Divisor	Number of Examples Having Zeros in the Quotient		
	Two-Digit Quotients	Three-Digit Quotients	
		Zero at End of Quotient	Zero in Midst of Quotient
2	18	180	162
3	27	270	243
4	36	360	324
5	45	450	405
6	54	540	486
7	63	630	567
8	72	720	648
9	81	810	729
Total	396	3960	3564

3,564 examples. Quotients like 100, 200, 300, etc., containing two zeros, are included in the numbers in the third column of the table, but not in the fourth. If they were included in both, the total of each column would be 3960.

There are, then, more examples having zeros in the quotient than we can hope to use. It is our responsibility to select from these 7920 examples those which will best develop the ability to deal with zeros in the quotients.

The fifth set of practice examples in division contains 34 examples having 0 as the second digit of a two-digit quotient. The divisor 2 is used twice; 3, three times; 4, four times; 5, 6, 7, 8, and 9, five times each. No remainder occurs twice for any one divisor. Each of the digits, 2 to 7, inclusive, occurs four times in quotients; 8 and 9 occur five times each.

Set VI contains 20 examples. Each example has a quotient of three digits, the last of which is 0. Practice is distributed over the remaining digits in divisors and quotients in much the same manner as in Set V.

Set VII includes 40 examples. Each has a three-digit quotient with a 0 in the second place. The quotients include the

PRACTICE EXAMPLES IN DIVISION. SET V

Zeros at End of Two-Digit Quotients

4/163	7/421	8/242	5/350	3/272
6/125	2/160	9/456	4/281	8/647
3/61	5/453	7/286	9/544	2/101
6/182	8/725	5/154	7/490	3/180
4/322	5/202	6/304	9/187	4/80
6/483	9/360	7/634	8/481	7/212
8/403	6/541	9/638	5/401	

PRACTICE EXAMPLES IN DIVISION. SET VI

Zeros at End of Three-Digit Quotients

6/4324	8/5044	5/4703	9/7291	7/5950
4/3883	7/1123	3/1680	8/3841	6/5583
5/3600	9/5760	2/1741	4/1401	7/6444
9/3334	5/901	8/2003	3/572	6/2762

digits 2 to 9, inclusive, nine times each, and 1, eight times. The larger divisors are given more intensive practice than the smaller, 2 being used twice and 9, eight times. Each of the possible remainders occurs approximately as frequently as it would in a chance selection of examples.

The amount of practice to be provided for each of the kinds of examples which have been discussed will, of course, depend upon the progress which the class makes. In the third grade we wish to fix well the fundamental division bonds, acquaint

PRACTICE EXAMPLES IN DIVISION. SET VII

Zeros in Midst of Quotients

7/2849	5/4033	9/2724	8/876	6/3617
9/4557	4/3231	7/6308	6/2434	3/2124
8/2422	9/6316	5/2049	4/1220	7/4261
6/5419	8/3261	2/1004	5/1505	3/1216
9/3686	7/1424	8/4866	6/4209	9/7240
4/2410	6/4830	9/2763	7/3546	5/1036
8/4011	9/8137	4/3605	7/6358	6/1838
9/1856	3/9029	8/3217	2/1417	5/2507

pupils with the technique of division (using the long division form) and master the zero difficulties, limiting our work to one-digit divisors and three-digit quotients. This will ordinarily require considerably more practice than is provided by the examples in our sets of practice exercises. These sets of exercises, however, will suggest how additional practice materials should be prepared.

In discussing division difficulties, we recognized zero difficulties only so far as they appear in the quotient. Undoubtedly, zeros in the quotient represent the most serious of the zero difficulties which pupils encounter in division. But zeros may also appear in the divisor and in the dividend. Zeros in the divisor will not concern us at this point since we do not consider two-digit divisors in the third grade. Zeros in the dividend, however, sometimes cause trouble. In examples like 8/408, the zero should cause little trouble for it is combined with the 4, and the pupil thinks, "How many 8's in 40?" But in the example,

7/2807, the pupil must actually divide 0 by 7. In examples such as this we have the zero facts concretely applied. Here we need to teach the fundamental principle that *zero divided by any number equals zero,* and apply this principle in a brief period of intensive practice. Perhaps a set of practice examples should be prepared which would provide for specific training on this form of the zero difficulties.

Checking results. Every division example solved should be checked. From the time of his introduction to the subject, the pupil should learn to apply the important principle that *divisor times quotient plus remainder equals dividend.* Or, when he has learned the signs, it may be written in equation form—divisor $\times$ quotient $+$ remainder $=$ dividend. In solving a division example, the pupil divides, multiplies, and subtracts; in checking that example, he multiplies and adds. Thus all four of the fundamental operations are practised with emphasis upon the two most recently learned.

The use of problems. We have frequently emphasized the importance of real and interesting problems as a means of applying the fundamental skills which are being acquired. One reason for the difficulty of division lies in the fact that pupils sometimes find it hard to see the usefulness of the process. In real life situations, problems involving division occur far less frequently than do problems in which the fundamental operation to be performed is addition, subtraction, or multiplication. But if interest is to be maintained at a desirable level, new phases of the division process must be introduced by real and interesting problems, and the teacher must search assiduously through the local environment, as well as through textbooks, courses of study and books for teachers, for good problems.

EXERCISES

1. Why should the first work in multiplication and division be given in close sequence?

2. How early would you give pupils their first acquaintance with fractions?

3. What is the relation of the fraction form to elementary work in division?

4. What are the advantages of teaching long division before short division?

5. Enumerate the division difficulties which were discussed in this chapter and illustrate each.

6. Should pupils be given as much practice on dividing 27 by 4, as 28 by 4?

7. Give directions for preparing sets of practice exercises on the primary facts of division with remainders.

8. When do pupils need higher decade subtraction?

9. Describe an introductory lesson which you would give on division with carrying.

10. In real life, what proportion of division examples have remainders?

11. If the divisor is 4, and the quotient is to have two digits, what is the smallest number which can be used as the dividend? The largest?

12. How many examples can we have if the divisor is 4 and the quotient is to have two digits? How many of these have no remainders? How many have zero in the quotient? How many involve carrying?

13. In teaching pupils to handle examples having zeros in the quotient, what different types of examples should be considered?

14. Why is it easier to overcome the zero difficulties if we teach the long division form for one-digit divisors?

15. When are zeros in the dividend hard?

16. How should pupils in the primary grades check division examples?

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Chapter X

THE SOLUTION OF PROBLEMS

In the preceding chapters, frequent reference has been made to the importance of problem solving in the primary grades. We have insisted that pupils should not only acquire skills in the fundamental operations but that they should also have opportunities to apply those skills to problem situations. The four operations are not ends in themselves but only means to an end; they are the tools which enable pupils to meet the arithmetic demands of life successfully.

Distinction between a problem and an example. Many teachers use the words "problem" and "example" synonymously. Many writers make no distinction between the two. It seems best, however, to limit the word "example" to arithmetical situations in which the operation to be performed is indicated. For instance, $\begin{array}{r}28\\ \underline{\times 4}\end{array}$, $\begin{array}{r}241\\ \underline{-163}\end{array}$, $28\overline{)6742}$, etc., are examples. It is not necessary that the signs, ×, —, etc., be present; a page of examples in one operation may be preceded by the word, "add," "subtract," etc. On the other hand, a problem is an arithmetical situation in which one must first decide what operation or operations to perform and then perform them. "Find the cost of 4 postage stamps at 2 cents each" is a problem, but $4 \times 2 = ?$ is an example. Every problem yields one or more examples.

Some may contend that this distinction is somewhat naïve. Long division is a real problem to the pupil who is just learning it, but, when learned, offers no more than a series of examples. What he must do in this subtraction example, $\begin{array}{r}600104\\ \underline{385647}\end{array}$, may be much more of a problem to a pupil than "What will 2 pencils

cost, at 5 cents each?" The distinction which we have set up is, in a measure, an arbitrary one, but it is useful. It is convenient to think of those arithmetical situations which are stated in words as problems and of those in which the operations are indicated as examples. This distinction has been made in all references to problems and examples in the preceding chapters.

Qualities of good problems. In selecting problems, the teacher should keep in mind certain qualities which good problems possess. Among these, we may indicate the following:

1. The problems must be real. That is, they must be such as actually occur, or might quite well occur, in the experiences of the pupils of the particular group for which they are selected.
2. Problems must deal with interesting situations. It is the business of the teacher to choose problems which children will want to solve.
3. Problems must be worded in interesting, appealing language.
4. In the wording of problems, the terms and expressions used should be those which the children can understand.
5. The problems should provide practice on the fundamental skills which the pupils have recently been trying to acquire.

Reality in problems. In Chapter I, it was pointed out that arithmetic was originally introduced into the curriculum as a practical subject, that the influence of faculty psychology was such as to give it an impractical content which, presumably, had high disciplinary value, and that the modern tendency is again toward the practical, or social. Widely used textbooks still contain much impractical material, however. The influence of the older disciplinary point of view can be seen today in most texts and courses of study. Teachers and school administrators seem to consider their pupils poorly prepared in arithmetic if they are unable to solve problems which, by their very nature, could not occur in a real situation. Examinations which have been pre-

pared for teachers still contain problems of the same impossible kind. For example, in a state examination in arithmetic, taken by teachers and prospective teachers in April, 1926, the following problem headed the list.

"A farmer sold two horses for $96.00 each. On one he lost 1/7 of the cost, on the other he gained 1/7 of the cost. (*a*) How much did he gain or lose by the transaction? (*b*) What per cent did he gain or lose?"

Assuming that the conditions of this problem actually occurred, one would, of course, have to know the cost of each horse, the loss on the one, and the gain on the other, before he could know that the loss and gain were each 1/7 of the cost. Since the answer had to be known to frame the problem, the problem is, by its very nature, impossible. It is an arithmetical absurdity.

There are many problems in our textbooks, our courses of study, the examinations which we set for our pupils, and the examinations which the teachers themselves must sometimes take which are unreal in the sense that the answers had to be known before they could be formulated. Problems of this category are less likely to occur in the primary grades than in the intermediate grades or the junior high school, but it would be well for the primary teacher to examine critically every problem designed for use in her class and to reject all whose answers were used in formulating them.

If the textbook which has been adopted for the use of the class contains unreal problems, the teacher should not hesitate to omit such problems and to replace them with better problems drawn from other sources. The following problems are typical of the kinds which recently published textbooks contain. Each states a situation in a form which is impossible or highly improbable.

1. Twenty boys went camping. All but 8 of them caught cold. How many boys caught cold?
2. What do you do with 11 and 5 to get 6?

3. I saw 956 boy scouts in a parade; 748 of them stood erect. How many were not erect?
4. There are 21 rose bushes in 3 equal rows. How many in a row?
5. Our electric light bill for February was $1.95. For May it was ⅔ as large. What was the bill for May?
6. John brought 3 books from the library and read 18 pages in each that night. How many pages did he read?
7. Henry is 6 years old, and Robert is 4 times as old. How old is Robert?
8. Francis has 7 dollars. George has 6 times as many dollars as Francis. How many dollars has George?
9. Mr. Harris drove his car an average of 228 miles a day on a 3-day trip. How far did he drive?
10. Earl paid 10 cents for a scout book and 4 times as much for a scout belt; how much did he pay for both?
11. Henry and John were reading the same story, and when Henry had read 58 pages of it he found that this was 25 more pages than John had read. How many pages had John read?
12. If there are 8 desks in each row and there are 40 desks in the room, how many rows are there?

One is much more likely to know the number of boys who have colds than the number who haven't. We do not do anything with 11 and 5 to get 6; we subtract 5 from 11 to find the difference or the remainder, not to get 6, unless the pupil first looks at the answer, if available, and then juggles figures to get it. Any one who could watch a parade of boy scouts, determine that the number was 956, and that 748 of them stood erect, would not find problem 3 very difficult (assuming that one would count the number of boys in the parade and the number erect, rather than the number whose shoulders drooped). Ordinarily, we should find the total number of rose bushes by counting the number in a row and the number of rows. Problem 5 is impossible, for the

electric bill for May would have to be known before one could determine that it was $\frac{2}{3}$ of the amount paid in February. John is an unusual boy if he read three books simultaneously to the extent that he read just 18 pages in each of the three the first night. The answers to numbers 7, 8, 9, 10, and 11 had to be known to frame the problems. The fault in number 12 is similar to that in number 4.

So well established is the practice of assigning unreal problems that many teachers who desire that their problems shall be real find it very difficult to detect all of the unrealities which they possess. We need to examine again and again the problems which we give children to solve if we would assure ourselves that they have a rightful place in the assignment.

Good problems deal with interesting situations. Many children like arithmetic. They like it because its tasks are specific, they can see what they have accomplished, and they know when a task is done. Some like it because of the puzzle element which it contains. Some, a few, will continue to enjoy it however fantastic and unreal the assignment may be.

Other children very heartily dislike this subject. They do not enjoy being puzzled by situations which, in themselves, seem of little moment. If they are interested in the subject it is only as a means to an end—the realization of some objective which they are unable to attain without computation.

For many children, probably the majority, the relationship between interest and reality is intimate. If problems are to be interesting they must be real. A few, usually the more gifted, will be keenly interested in the unreal problem but they will enjoy the real problem as much, provided that it is as difficult and equal to the unreal problem in other respects.

If problems are to be interesting, they must not only be real but must also deal with situations which are real in the experiences of children. Teachers and textbook writers have been very careless in this regard. They have provided problems which not only neglected the interests of children but which disregarded

the interests of all groups, except, perhaps, the interests of those who made the problems. Consider the following four problems:

1. A box has 8 corners. How many corners have 5 boxes?
2. Washington was born in 1732 and died in 1799. How many years did he live?
3. A fly has 6 legs. How many legs have 7 flies?
4. One wall contains 5,460 bricks. Another wall contains 2,545 bricks. How many more bricks in the first wall than in the second?

Problems showing this utter disregard for the interests of children are easily found in books in wide circulation today. Authors whose reputations are world-wide are guilty of placing before children problems so uninteresting that they should never be found in the schoolroom. Consider these:

1. If Mr. Smith uses 325 pickets for a fence on one side of his lot and 268 pickets for a fence on another side, how many pickets does he use for both sides?
2. A grocer sells 189 oranges one day and 226 the next day. How many does he sell in two days?
3. A man bought two second-hand automobiles. For one he paid $235 and for the other $275. How much did he pay for the two?
4. If a man has $375.40 in the bank and draws out $139.75, how much money has he left in the bank?

Surely, we can do better than these, in our efforts to break away from the unreal and uninteresting problems which the older methods employed. These four problems may be interesting to certain adults, although one finds it difficult to imagine Mr. Smith estimating his needs for pickets so exactly, or the grocer keeping such accurate account of his daily sales of oranges, but for children in the third grade, they have little, if any, appeal.

The wording of problems. Much of the success which children attain in solving problems seems to depend upon the language in which the problem is expressed. Textbook problems are

too often worded in a dry, very concise manner and make little appeal to the imaginations of the children. Well-worded problems, on the other hand, read like stories to children and have a strong imaginative appeal. Well-worded problems, consequently, are usually not briefly worded but go to some length to make the details of the situation clear, vivid, and interesting. As Miss Wilson expresses it, "The first step in teaching children to read problem-material is one of making the story in the problem so real and so vivid that the pupil realizes the situation." [1]

Doctor Myers has made an experimental study of "how problems designed to stimulate vivid imagination compare in difficulty with very similar problems of the dry, concise, traditional sort." [2] He presented 12 problems to 486 pupils. Six of these were worded in the usual formal fashion and six others were designed to have an imaginative appeal. Each formal problem was intended to be equivalent to a corresponding imaginative problem so far as the intrinsic arithmetical difficulty was concerned. In five of the six cases, the imaginative problem was solved correctly by a larger number of pupils than was the corresponding formal problem. In the sixth case, the number of correct solutions was equal for the two problems.

We give below four of the 12 problems which Myers used. Two of these are formal problems and two, imaginative. Each imaginative problem follows immediately after the corresponding formal problem. That is, A1 and B1 are the formal problems and A2 and B2, the imaginative problems.[3]

A1. Laura, whose grandmother gave her $1.50, saved 5 cents each school day from her lunch money. How many school days before Laura will have enough money to buy a book which costs $2.50?

A2. Little Betty wanted to surprise her mother on Christmas.

[1] Wilson, Estaline. "Improving the Ability to Read Arithmetic Problems." *Elementary School Journal,* XXII: 380-386, January, 1922.
[2] Myers, Garry Cleveland. *The Prevention and Correction of Errors in Arithmetic.* The Plymouth Press, 1925, p. 65.
[3] Myers, *op. cit.,* pp. 65-67.

She knew that her mother needed a good pair of scissors. The old ones would not cut well. One day Betty went with her daddy to a big store to see what a good pair of scissors would cost. She saw one for $2.00. Betty had 95 cents which her grandmother had given her. "I shall save 5 cents every day from my school money." After —— school days she bought the scissors, which lay at the foot of the Christmas tree on Christmas morning marked, "To my dear mother, from Betty."

B1. After traveling 160 miles, a man has 4 gallons of gas left in his automobile. How many miles did he get to a gallon of gas if he bought 8 gallons on the way and had 6 gallons when he started?

B2. Last summer Agnes Purdy, her brother Archie, and their parents took a trip in their Ford. Archie measured the gasoline when they started. "We have eight gallons," he told his father. At the end of the day he found 4 gallons of gasoline in the tank. They had bought 6 gallons at a station on the way, and had traveled 150 miles. Agnes told her mother that they had made —— miles to a gallon that day.

The number of correct solutions for these four problems was as follows:

A1	181	A2	320
B1	197	B2	253

Or, expressed in percentage form, A1 was solved correctly by 37 per cent of the 486 pupils, and A2 by 66 per cent; B1 was solved by 41 per cent of the pupils, and B2 by 52 per cent. The average per cent correct for the two imaginative problems was 59; for the formal problems, it was 39.

These four problems are not offered as suitable problems for the primary grades, but are intended to illustrate the types with which pupils are more likely to be successful. These problems are too difficult for use in grades one to three. Myers tested pupils in the fifth and higher grades only.

Further suggestions concerning the wording of problems have

been given in previous chapters. (See pages 53-55, 142-144, 160-161, 186.)

Avoiding difficult terms. Not only should problems be so worded as to have an imaginative appeal, but very difficult, unusual words and technical terms and forms of expression should be avoided. We shall, eventually, want to provide that our pupils become familiar with many new words and terms; but when a new process is being learned, or a new type of skill is being developed, extraneous difficulties should not be allowed to creep in and block the pupil's progress. The writer has known pupils to fail in their efforts to solve a problem because of the presence of a single word whose meaning they did not know; when this word was explained, or an easy synonym substituted, they were able to proceed with the solution. Frequently, however, problems contain many words which are unknown to the children.

Thorndike discusses language difficulties in arithmetic problems and gives a list of words [4] used in books designed for pupils in the second and third grades. The list includes *absentees, admitted, alternate, barley, bronze, charity, dictation, enrolled, generally, indicated, installments, margin, mineral, opposite, proprietor, rather, rye, suggested, syllable, transaction, various, worsted,* and many others of equal or greater difficulty. It is doubtful whether one-fourth of the pupils in the grades for which the problems were intended could read these words. The use of words such as these blocks the progress of the pupils. One might almost as well state the problems in a foreign language.

In a certain schoolroom the pupils recently read, or tried to read, the following explanation of the meaning of *dozen*. "How many cubes make a half dozen cubes? How many cubes make a quarter of a dozen cubes?" Compare the arithmetical difficulty of recognizing and understanding that there are 6 things in a half-dozen with the linguistic difficulty of the word, *cube.* Why not say *blocks?*

[4] Thorndike, Edward L. *The Psychology of Arithmetic.* The Macmillan Company, 1922, pp. 90-91.

Examine the following problems:

1. Harry walked 462 ft. to a certain telegraph pole and then walked 464 ft. farther to a watering trough. How many feet did he walk in all?
2. Henry bought a pair of tennis shoes for $1.85 and gave the dealer $5. How much change was due?
3. 38 passengers were riding in a street car. At a transfer point 16 got off. How many remained in the car?
4. The World's Fair was held in Chicago in 1893. The Centennial was held in Philadelphia in 1876. How many years passed between the two exhibitions?
5. The original group of states in the United States numbered 13, but now there are 48 states. How many states have been added?
6. How old is a man who is 4 score and 10 years?

These problems are typical of those which may be found in the third grade portion of recently published textbooks. We wonder how many third grade pupils who could understand the conditions of the problems if stated in easier language and who could perform the necessary computations would be baffled by the words, *telegraph, trough, tennis, dealer, due, passengers, transfer, Centennial, exhibitions, original, group,* and *score.*

A teacher may judge fairly accurately the importance and difficulty of words by looking them up in Thorndike's *The Teacher's Word Book,*[5] a book which gives in alphabetical form the 10,000 commonest English words and indicates the importance of each. Of the 13 words mentioned in the preceding paragraph, three are found to be in the second thousand; they are, *due, group,* and *score. Passenger, Philadelphia* and *original* are in the third thousand; *telegraph, dealer,* and *transfer* are in the fourth thousand; *trough* is in the fifth thousand; *tennis* and *exhibitions* are in the sixth thousand; *Centennial* is not found in this list at all.

[5] Thorndike, Edward L. *The Teacher's Word Book.* Teachers College, Columbia University, 1921, 134 pp.

Chase undertook an investigation to show what certain words which frequently occur in arithmetic texts mean to children.[6] The children were given a list of 47 words, preceded by these instructions:

1. Put *w* beside each word that tells what a man's work is.
2. Put *m* beside each word about money.
3. Put *l* beside each word that might be used about land.
4. Put *i* beside each word that is the name of something to put things in.

The list of words with the per cent of pupils in the fourth and fifth grades who did *not* indicate the correct meaning follows. Miss Chase's list as originally published has been rearranged and alphabetized.

GRADE	IV	V	GRADE	IV	V
acre	56%	32%	gardener	17%	8%
area	91	36	income	96	41
attend	4	4	insurance	91	42
barrel	30	32	jars	40	16
basin	26	20	lot	9	12
basket	13	8	machinist	35	28
bin	43	20	mason	78	36
broker	56	45	merchant	45	24
bucket	30	16	miller	26	12
building lot	65	32	nickel	13	16
carpenter	26	8	owe	91	60
cashier	52	20	pasture	52	32
cistern	100	88	poultry	40	36
coins	26	16	profit	78	45
collect	17	8	real estate	100	68
commission	100	68	rent	56	28
customer	35	44	retail	30	16
dealer	35	28	salary	40	20
debts	91	32	schedule	17	4
earn	60	36	tailor	13	4
excavate	4	4	tank	65	36
expenses	65	20	teamster	88	40
fares	65	60	wages	70	28
field	9	8			

[6] Chase, Sara E. "Waste in Arithmetic." *Teachers College Record,* XVIII: 360-370, September, 1917.

Unfortunately, no data are given for the third grade. We can infer, however, that the per cents for the third grade would be higher than those given for grade four. It will be noted that the word *dealer* which we found in a problem, and which Thorndike lists in the fourth thousand according to frequency of occurrence, was unknown by 35 per cent of the pupils of the fourth grade.

Select Problems which practise recently developed skills. Problem solving is the primary objective in arithmetic instruction. Skill in the fundamental operations is but a means to this end. Yet so important is it that pupils become proficient in the operations in the primary grades, definite efforts should be made to apply these skills in real problem situations. In deciding upon the objectives of the arithmetic course of study as a whole, we determine the skills which are to be acquired by analyzing the problem situations which pupils must meet. Thus, in the large, the fundamentals are taught only as a means to an end; but, in the primary grades, we may temporarily and in part make problems a means to the end of developing adequate skills in the fundamental operations.

From time to time in the preceding chapters we have suggested that the pupil's skills be applied to problems as soon as acquired. Thus, when the pupil has learned the first group of addition and subtraction facts, he will be given interesting problems in which he will have an opportunity to see that the facts which he is acquiring are really useful facts. In this way interest may be maintained and progress assured.

We have also suggested that each new type of difficulty will be introduced by a problem in order that the need of mastering the difficulty may be made apparent. Then, when the difficulty has been explained and the pupil has mastered it fairly well, additional problems of the type first used will be given for practice.

Three classes of problems. The problems which children are given to solve may be conveniently divided into three classes. First, we have those which are based upon situations which are

actually present to the pupil's senses. When the pupils engage in some sort of coöperative enterprise, such as playing store, laying out a baseball field, planning a school garden, deciding which side won in a game or contest, etc., the problems arising may be said to be first class problems. Because of the realistic appeal which problems of this class make, pupils will attack them with greater enthusiasm and will be more successful in solving them than if the problems are drawn from the affairs of others. They are less likely to engage in purely random manipulations, or to add all of the numbers because there are three or more of them, or to divide because they obtain a quotient with no remainder.

Second, there are those problems which are based upon situations which are not actually present to the pupil's senses but which he can readily imagine. When a pupil makes his future plans, or lives in his imagination the experiences of others whom he reads about or hears about, the problems which arise will make almost as strong an appeal as though they were drawn from his actual present or past experiences. Pupils plan for parties, Saturday excursions, and camping trips, they estimate how long it will take them to save enough money to buy some toy or other article and they listen with interest to the accounts of the experiences of their friends and other associates; good problems arise out of such experiences. Problems based upon hypothetical hikes, camping trips, automobile tours, fishing trips, etc., are problems of the second class. They cannot be first class problems unless the pupils actually take the trip, make the hike, etc., but because of the resemblance of the conditions to experiences which the pupils have had, and because of the interest which they have in such experiences, they easily imagine the conditions assumed in the problems and delight in solving them.

The third class of problems includes those which are mere verbal descriptions of situations which the pupil does not readily imagine—problems which are based upon situations described in the words of others. Since the situations assumed in prob-

lems of this class are not real situations to the pupils and since they are not situations which they can readily imagine themselves experiencing, they will have only a partial understanding of them, they will be slightly, if at all, interested in them, and will, as a consequence, make poor progress. Many of the difficulties which both pupils and teachers experience in learning and teaching problem-solving can be charged to the use of problems which are neither real nor interesting, problems which are mere verbal descriptions of situations and conditions foreign to the interests and experiences of children. Pupils are much less likely to be successful with problems of this class. If they are successful, little is gained for life problems are not problems of this kind.

These three classes of problems have been stated in the order of their value. The first class is the best and the third class is the poorest. But this is the reverse of the order of the frequency of their occurrence in many textbooks. Unfortunately, in many of the arithmetic texts which have been written for children, the bulk of the problems are clearly problems of the third class. This fact alone would seem to account for much of the failure of pupils in problem solving.[7]

Textbooks can hardly be expected to provide first class problems. Problems which are real to one child or group of children cannot be better than second class problems to another pupil or class, unless the latter have had identically the same experiences as the former. First class problems will have to come from the immediate environment and experiences of the pupils of the group and will be initiated by the pupils and the teacher. We can, however, expect our texts to contain an abundance of problems of the second class—those which make a strong appeal to the imaginations of children. The desirable text will strive for the complete elimination of problems of class three, will contain an abundance of problems of class two, and will offer many suggestions and hints for the use of problems of class one.

[7]See Thorndike, Edward L. *The New Methods in Arithmetic.* Rand McNally and Company, 1921, pp. 126-132.

Training pupils to solve problems. If we limit our discussion of the teaching of problem solving to the primary grades, there is little to be said in the way of developing a definite teaching technique. Scientific studies of the factors conditioning a pupil's success in solving problems indicate that his success seems to depend chiefly upon his intelligence or his general native ability, his skill in silent reading, and his skill in computation. This means that the teacher must select and use problems which are within the pupil's grasp, must see that they are expressed in language which he can understand, and must provide that he has developed the fundamental skills necessary to their solution.

It frequently happens, however, that a pupil will find a problem interesting, will understand its conditions and will see what he is to do, but will be unable, at first, to plan a solution. In such cases, it seems best to provide for a systematic attack. Thorndike suggests that the pupil be taught to ask and answer these questions, if he is not sure that he knows how to proceed.[8]

1. What am I to find out?
2. What facts (numbers) are given?
3. What shall I do with the numbers?

Answering the first two questions does not guarantee success with the third but it seems to increase the probability of such success. Sometimes, if the problem is long and rather involved, it is well to have the pupil write out his answers to the first and second questions to assure that he is clear in his thinking about them. If the problem permits a simple diagrammatic representation, on which the facts or numbers secured in answer to question two are written, the pupil will frequently see more readily how to proceed with the solution. The suggestions of this paragraph, however, are of greater value for the intermediate and upper grades than for the primary grades.

No effort should be made to train pupils to record solutions in systematic and prescribed form, in the primary grades. It is

[8] Thorndike, Edward L. *op. cit.*, p. 139.

enough that they be interested in the problems and enjoy the thrill which comes from success in solving them.

When a pupil has difficulty with a problem the teacher's richest opportunity presents itself. She will first assure herself that the conditions of the problem are understood, that there are no words or expressions which the pupil does not understand and that he possesses the fundamental skills necessary to do the computations which the problem calls for. Then by skillful questioning she will lead him to devise a plan of solution for himself. In this case, telling is not teaching. She will invent other problems which are much simpler and much easier and then lead him to see the analogy between them and the problem with which he is having difficulty. By diagnosing accurately his trouble and by using all of the information which she has concerning his abilities and disabilities, she will be able, if she is a good teacher, to stimulate him to think for himself and will leave him a better thinker as a result of the experience.

More detailed suggestions for teaching children to solve problems are deferred for consideration in the intermediate grades.[9]

The use of projects. The primary teacher will derive many problems from a single project with which the pupils are engaged. The project method of teaching is not discussed in this book for it is larger than any one subject in the curriculum. But the effective accomplishment of a project is certain to involve many problems, affording not only a motive for learning to solve problems but also an excellent opportunity for practising the skills which have been acquired.

In projects and independent of projects, problems will frequently be grouped so that several are derived from a single situation. Thus, a continuous narrative will run through a series of problems pertaining to one situation or to the experiences of one child or group of children. An illustrative example will be found in Chapter IV. (See pages 54-55.)

[9] See the author's *Teaching Arithmetic in the Intermediate Grades* Silver, Burdett and Company, 1927, Chapter X.

EXERCISES

1. What is the difference between a problem and an example? How may an example be a "problem"?

2. Enumerate the qualities of good problems.

3. Is this a real problem? "Harry lost half of his marbles and had 12 left. How many had he at first?"

4. Can a problem be real if the maker had to know the answer to make it? Why?

5. How may a problem be real and yet be uninteresting?

6. Do you believe that future textbooks for the primary grades will be larger than they have been in the past? Why?

7. Summarize Myers' experiment on the use of imaginative problems?

8. A teacher read the following problem and said, "That would be all right for the primary grades because it tells that a flageolet and a violin are musical instruments." What is your opinion?

You see a flageolet and a violin. They are musical instruments. One musical instrument and one musical instrument are how many?

9. Summarize Miss Chase's study of vocabulary difficulties in arithmetic.

10. Name the three classes of problems as given in this chapter. Illustrate each.

11. What suggestions can you give for training pupils to solve problems?

12. In what way do problems arise from projects?

13. Prepare a list of five problems all based upon the same general situation, and suitable for pupils of the second grade.

14. Prepare a similar list of five problems for use in the third grade.

15. Point out the fault in each of the following problems.

(*a*) One train had 7 cars and another train had 15 cars. How many more cars did the latter have than the former?

(*b*) If a horse trots 9 miles in one hour, how far will be travel in 10 hours?

(*c*) In this picture you see one kitten on the ground and one kitten on the stump. If you should ask me how many kittens are on the stump, what would be my answer?

(*d*) Jack's father gave Jack one half of 12 dollars, and gave Jack's sister one fourth of 12 dollars. How many dollars did his sister receive?

(*e*) Madeline saved 12 dollars. If she gave one fourth of it to the Salvation Army, how much did she give the Salvation Army?

(*f*) An office building having 45 windows requires 4 sashes for each window. How many sashes will be needed?

(*g*) The sides of a square are all of the same length. Find the length of one side of a square field if the distance around it is 180 feet.

(*h*) A man divided 480 toy soldiers equally among his four boys. How many soldiers did each of the boys receive?

(*i*) Edmund and his father drove their car 456 miles in 4 days, running the same number of miles each day. How many miles did they run each day?

(*j*) Jack helped his father plant 6 rows of potatoes, and they planted in all 1182 hills. How many hills were there in each row?

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CHAPTER XI

THE COURSE OF STUDY

Modern tendencies. In the modern school system, much thoughtful attention is given to the construction, revision and use of the course of study. Whereas, a few years ago, the course of study in a small city system of schools was often the product of one man, the superintendent of schools, who frequently employed the scissors and paste-pot method in constructing it, the course of study in a modern system of schools is the result of the joint effort of the members of a committee composed of classroom teachers, principals and supervisors, who have devoted a year or more to the development of a course in a single school subject.

For example, the Denver course of study in arithmetic for grades one to six, inclusive, went to press in the summer of 1924 as the result of the labors of a committee of seven who had worked under the direction of two curriculum specialists, faculty members in nearby educational institutions of collegiate grade. The committee members had met and worked on school time, substitute teachers taking their places in the schools. The committee had the advantage of a good professional library supplying the best of current expert opinion on the teaching of arithmetic. When their work had reached the proper stage of development, leading specialists were brought to Denver to consult with and advise the committee; later these specialists criticized the manuscript which the committee prepared and offered suggestions for improvement. The result was a document of 228 pages for the six grades of the elementary school.

Any course of study at all is, perhaps, better than none. But if classroom teaching is to be on a high plane, a course of study indicating more than the mere textbook pages to be covered is necessary. A good course of study will make definite references to the material of an adopted text, and, in addition, will set forth the specific aims to be accomplished in each grade, will discuss general methods of teaching the subject as a whole as well as specific methods for each topic, will offer suggestions on adapting instruction and materials to local conditions, will offer tentative standards of attainment for each grade in each subject, and will make provision for individual differences found among children, setting forth the minimum essentials for slow children and an enriched course for the more gifted pupils.

Continuous revision necessary. A good course of study is not a static thing but must ever be revised to incorporate better materials and better methods of instruction. As has been indicated in preceding chapters, there are topics in arithmetic toward which a very tentative attitude must be taken as regards the organization of materials and the methods of instruction. For example, no one knows which is the best method of teaching subtraction; but some day we hope that experimental studies will indicate a clear and distinct advantage of one method over others. In the meantime, the best that a group of teachers can do is to examine the literature on the subject, which is largely theoretical, and adopt, tentatively, some one method, possibly letting the prevailing practices of the neighborhood be the principal factor determining the method adopted. But when this question is settled by experimentation, if it ever is, the course of study should be revised accordingly.

Adapting the course to individual differences. Pupils differ tremendously in the extent to which they can profit from instruction. In a sixth grade class it is not at all unusual to find pupils who fail to reach the median scores of fourth grade pupils in a standardized test, and others who exceed the median scores of the eighth grade. A typical second grade class, composed of

pupils whose average chronological age is, say, seven and one-half years, will contain some pupils whose mental ages reach ten years or higher, and others whose mental age is five and a half, or lower. A range of mental ages in the second grade from five to ten presents an extremely difficult problem for the teacher to handle.

Ordinarily, the teacher undertakes to adapt her instruction to the average ability level found in her class. The majority will get along very well with such instruction, if it is skillful, but the extreme deviates will gain little, if any, advantage from such teaching. The dull will be unable to keep pace with their classmates and, as they lag behind, will become more and more discouraged with their progress. The gifted will find the assigned tasks easy, will probably be bored with explanations which they do not need, and may seek other means of amusing themselves.

The modern course of study will make very definite provision for individual differences. In the first place, it will provide that the class be sectioned into groups; there will usually be three—a slow group, an average group, and a rapid group. The slow group will include pupils who cover less than a year's work in a year's time but who do their work well as far as they are able to go. The rapid group will contain pupils who will eventually be accelerated in their progress through the grades, completing the six grades of the elementary school in five, or possibly four, years.

In the second place, there will be differences in the material assigned to these three groups. The slow group will cover just the minimum essentials, while the rapid group will have supplementary assignments, including topics and methods of solution beyond those ordinarily included in the course of study. This method of providing for individual differences will, however, have little, if any, application to the course in the primary grades. In the primary grades, only those facts and skills which everyone must acquire will find a place in the course of study.

In the third place, methods of instruction will be modified to

suit the abilities of the pupils in the various groups. The slow group will require explanations in the simplest, most concrete form and will need to have these explanations frequently repeated, varying the approach and the nature of the illustrative material. The bright group, on the other hand, will see reasons and will understand processes more quickly.

In teaching slow groups, it is sometimes necessary that the "why" of a procedure be omitted entirely and that the pupils be allowed to devote themselves entirely to the "how"—to gaining skill in the procedure. In teaching carrying in addition, for example, it is sometimes best to simply tell dull pupils what to do when the sum of a single column is a number of two digits and offer no reasons for our recommendations. The reasons should be supplied, of course, if the pupils are able to comprehend them.

THE COURSE FOR THE FIRST GRADE

We have already devoted a brief chapter to a discussion of the place of arithmetic in the first grade (Chapter II). In that chapter an effort was made to state the trend in recent years as regards the amount of formal and incidental instruction in arithmetic in grade one and to summarize such evidence as had been collected on the extent to which arithmetic instruction should be included in this grade. The only definite statement as to what should be included in the first grade course was contained in a quotation from a bulletin of the Bureau of Education, entitled, *A Kindergarten-First-Grade-Curriculum,* prepared by Miss Alice L. Harris.

Why arithmetic should be included in the first grade. The writer is ready to agree that if arithmetic is entirely omitted from the curriculum of the first grade, pupils will readily overcome this apparent handicap and, a year or two later, will have covered as much subject matter as they might have covered if the subject had been taught in grade one. However, the first grade pupil's out-of-school needs for number are so numerous and so urgent

that there is grave danger that he will form undesirable habits in arithmetic due to his own efforts to meet number situations for which he is not prepared, or, due to unskilled instruction which he may receive at home. It seems best, then, to make a definite place for arithmetic instruction in the first grade but to limit the subject matter to that which the majority of the pupils will have a real need for in their out-of-school experiences.

It will be apparent that any specific recommendation as to the items which should be included in the course for any grade must be more or less arbitrary. The recommendations which follow have been made after a careful consideration of such scientific evidence as is available, after studying the opinions of primary supervisors, teachers, and other expert students of primary education, after reading the courses of study of many of the more progressive school systems in the United States, and after reading two dozen or more books on the teaching of arithmetic.

The course in outline. We shall present an outline of the course suggested for the first grade under six topics: (1) counting and reading and writing numbers; (2) addition; (3) subtraction; (4) problems; (5) measurement; (6) activities.

1. Counting and reading and writing numbers.

(*a*) Count by 1's, 5's, and 10's to 100 and by 2's to 20.
(*b*) Read and write numbers to 100 or higher.

Most children are able to do some counting when they start to school. Many are already able to count to 100 by 1's, 5's and 10's. This counting is usually largely rote counting, learned in games, as hide-and-seek. The chief purpose in the first grade instruction in counting is to bring all to this minimum level of attainment so that they may have a common background of counting experience upon which certain more formal work of the second grade may be based and to see that all are able to meet the number needs which arise in connection with other activities both in and out of school. Counting by 2's is less commonly known by children entering the first grade. It is easily learned,

however, and is useful as a more rapid method of counting objects and as a means of counting children as they march two abreast, etc.

We would have all first grade pupils taught to read and write any one-digit or two-digit number. The extent to which any first grade class will be taught to read and write numbers will depend largely upon their needs for these skills in other school work, particularly reading page numbers in books which they use. Primers and other first grade readers usually contain 100 or more pages. The Berkeley, California, Course of Study in Arithmetic, 1923 edition, for example, recommends that pupils in Grade 1B be taught to read and write numbers to 120, the number of pages in the State Primer, and in 1A to 132, the number of pages in the State First Reader. To cover all such needs, some courses of study recommend that pupils in the first grade be taught to read and write all numbers to 200. It seems better, however, to say "100 or higher" and let the actual limit be set by the number of pages in the books used as texts.

First grade pupils have greater need for reading numbers than for writing them; we might then, set a lower goal for the latter than for the former. The more general practice, however, is to carry instruction in the two together. This probably means an economy in learning.

Four-digit numbers when they mean amounts of money, as $23.75, will be read as two two-digit numbers, one for the dollars and the other for the cents. Thus, the pupil will read $23.75 as "23 dollars and 75 cents," being taught to say "and" where the point is. Children read such numbers in shop windows on price tags. Telephone numbers and automobile license tag numbers will be read by calling the single digits. (See page 24.)

2. **Addition.** We recommend that the 25 addition combinations whose sums do not exceed 10 be taught in the first grade. (See page 33.) Some schools teach the entire 45 combinations in this grade; others teach the 33 whose sums do not exceed 12. Many progressive school systems, however, give no work in addi-

tion in this grade. We have decided to include these 25 combinations in the first grade course because the out-of-school uses of arithmetic by first grade children very frequently involve addition. Miss Smith's study, it will be recalled, showed that addition was the most frequently used process among such children, occurring more often than counting and reading numbers combined. (See page 12.)

If the teaching of these addition combinations means interference with the teaching of reading we should be quite willing to postpone all work in addition to grade two, but we believe that most first grade teachers will be able to follow the program which we are setting up without interference with a strong reading program and with advantage to the pupils. The chief advantage to the pupils lies in the avoidance of the danger that they will learn to add through counting before instruction in addition is definitely taken up in the school.

3. **Subtraction.** We have pointed out in Chapter IV that the addition and subtraction facts should be taught together. Since we are including the 25 addition combinations whose sums do not exceed 10, the 25 corresponding subtraction combinations will be taught also. This means that 90 of the 162 primary addition and subtraction facts will be taught in this grade. Forty-five of these are addition facts and 45 are subtraction facts.

Miss Smith's study showed that subtraction ranked third in the list of 10 processes in the lives of first grade children. Addition, counting, and subtraction together made up 70 per cent of such uses.

4. **Problems.** Problems in which the operations to be performed are limited to the 90 addition and subtraction facts taught in this grade will be used frequently in first grade classes. The work will be oral, except the writing of the numbers involved; near the end of the year, however, problems worded in language which the pupils have learned to read may be presented in printed or mimeographed form. They will be the simplest one-step problems. The nature of the situations involved, the wording of the

problems, and other points have been discussed in the preceding chapter. (See also pages 53-55.)

5. **Measurement.**

(*a*) Time: day of the week, month, and day of the month, yesterday, tomorrow.
(*b*) Length: inch, foot.
(*c*) Liquid: pint, quart, half-pint (2 glasses or cups make one pint).
(*d*) Coins: cent (penny), nickel, dime, quarter. The signs, \$, ¢.
(*e*) The terms, dozen and half-dozen.
(*f*) Fractions: the meaning of "one-half."
(*g*) Comparisons: taller, shorter (longer, shorter); larger smaller; higher, lower; heavier, lighter; nearer, farther; more, less.

Children have early need for learning the days of the week—their names and their sequence. We make our plans, in school and out of school, by referring to the days by their names. We do not have school on Saturday or Sunday; Thanksgiving falls on Thursday; Eleanor's birthday is next Tuesday; etc. The words "yesterday" and "tomorrow" will be learned in similar fashion. The children read days of the month on the calendar, learn when their birthdays and the birthdays of relatives and friends come, find on what day of the week certain dates fall, etc. They learn that school opens in September and closes in June; that Valentine day comes in February, Thanksgiving in November, Christmas in December, etc. Pupils gather information regarding time (as they do regarding other topics in measurement) entirely incidentally or as a by-product of other activities and interests.

Each pupil should own a foot ruler and learn concretely through measuring the meaning of inch and foot and that there are 12 inches in a foot.

Where children drink milk at school, it is usually delivered in half-pint bottles, a bottle making a glassful. At home, they see deliveries of cream and milk made in half-pints, pints and quarts.

Actual experience leads gradually to an appreciation of the relative sizes of these measures.

Through earning, saving, and spending their own money, through depositing in the school bank, and through playing store, pupils become acquainted with the four coins named. A partial appreciation of the relative values of these coins should be gained through acquaintance with the fact that five pennies are equal to a nickel, that two nickels are equal to a dime, and that a quarter is 25 cents. The signs $ and ¢ are frequently not given until later, but they are included here because pupils need them in reading price tags when playing store and in reading actual price tags in shop windows.

In hearing his mother order groceries, the pupil frequently hears "a dozen oranges," "a half-dozen eggs," "a dozen bananas," etc. In school, he hears "a dozen readers," "a half-dozen pencils," "a dozen pens," etc.

The meaning of "one-half" should be made clear through objective illustration. Many young children think of a half only as a part. In dividing candy with brothers and sisters, they will frequently give a half to each of three or more persons! Through the use of apples, pieces of paper, and the like, we shall teach them that to get one-half, we must divide the object into *two* parts and that the two pieces must be of the *same size*.

Comparisons will be made between the heights and weights of two children, the size and weight of books and chairs, distances, amounts of money, etc. There will be many opportunities for making these comparisons. Refined measurements and more exact comparisons will be made in later grades; in the first grade it is sufficient to be able to tell which of two is the larger, longer, more, etc.

6. **Activities.**

(*a*) Comparing sizes, ages, numbers, etc.
(*b*) Counting pupils, materials, etc.
(*c*) Keeping score in games.
(*d*) Keeping various room records.

(*e*) Playing store.
(*f*) Buying and selling the school milk.
(*g*) Buying stamps at the school bank.
(*h*) Finding pages in books.
(*i*) Measuring lengths of various objects.
(*j*) Conducting a "toy shop."
(*k*) A doll sale.
(*l*) Our flower shop. Etc.

The list of activities can be made quite long. Those given have been tried and found good. Each of the proposed activities will suggest others to the resourceful teacher. Further suggestions will be found in the references at the end of this chapter.

Most of the activities suggested do not require comment; their names suggest their nature. A more detailed description will be supplied for the last three, however, since they are not so well known. Two of these have been taken from *The Twentieth Yearbook of the National Society for the Study of Education,* a detailed reference to which will be found in the references at the end of the chapter.

"A Toy Shop"

(Reported by Alison Hayne, Detroit, Michigan)

"The shop was constructed from materials borrowed from the kindergarten. The children made toys, dolls, doll clothes, beads, doll house and furniture, picture books, blocks, animals, picture puzzles, clay dishes, vases, etc. The arranging and selling of toys was also done by the children. Their plans and discussions formed the basis for board reading lessons. The buying and selling involved the use of numbers. The prices were marked from five to twenty cents." (p. 18.)

"A Doll Sale"

(Reported by Marguerite DeLano and Myrtle L. Kaufmann, Springfield, Illinois)

"It was desired to raise money enough to buy rubber tips for the chairs in the first grade room. The children suggested means

of raising the money and finally decided to sell paper-bag dolls to children in other rooms in the building.

"In making the dolls cotton was put in the bottom of the bag and tied to make the head. The top of the bag, when inverted, served as a skirt stiff enough to support the doll in a standing position. A face and hair were painted on the head. Variation in facial expression made each doll different from the others. Rolls of paper put through slits in the dress and fastened with paper fasteners formed the arms. Dresses and caps were made of colored crêpe paper. The variety of dolls made included: witches, Red Riding Hood dolls, sun-bonnet babies, and dolls with and without aprons.

"The children were in charge of the sale. Five cents was the price of each doll. Advertising posters were planned and made.

"The plan called for:

1. Language.
   - (*a*) Oral: discussion of plans and the announcing of the sale in other rooms.
   - (*b*) Written: addressing of package sent out of city to a visitor who had left an order.
2. Reading of posters.
3. Art: Selection of pictures for posters, and arrangement of same. Cutting of letters for posters. Decorations of doll heads. Planning color combinations for the dresses.
4. Occupation work: Preparation for making the dolls was given in the period set aside for project work. Later children were able to work unsupervised at tables and workbenches. After the sale the children made dolls for themselves.
5. Number: The class felt the necessity of learning to make change.

"As a result, every chair in the room is rubber-tipped." (pp. 22-23.)

The next activity, "Our Flower Shop," is especially good for use in the first grade.

"Our Flower Shop"

(Reported by Miss Jeanette Smith, Training School, Ohio University)

"Late in February the children (first grade) suggested that we have a store where we could sell things. After a visit to the city flower shop, the children decided we should have a flower shop. Through our trip they had gained many ideas of what a flower shop should contain. The children also suggested that our library could be used for our flower shop if we could find another place for our books. As a result a fresh coat of green Fresco was applied to our house and the shelves soon contained vases, baskets and flower pots, all made by the children (except the flower pots) and painted by them. A counter was made also. Bulbs were planted in water and seeds in good soil brought from the University greenhouse in our first-grade wagon (made by two of our boys). Labels were made for the lettuce and radish seeds. Posters were made to announce our sale to all grades in the building. For these, pictures were cut or painted and statements were printed as,

For Sale, Buy Seeds Here, Plant a Garden, First Grade Flower Shop, Vegetable and Flower Seeds, 1¢, 2¢, etc.

"We wanted our mothers to come, so we wrote invitations inviting them. We wished to tell them all about our shop, so we wrote stories telling just what we had been doing. These were typed and bound by the children into books. With these stories, some spring poems and songs, we entertained our mothers on the day of the sale.

"We believe we gained more in interest and fun through selling things than from any other part of our project. We made price tags for our seeds, vases, pots, etc. We then decided that only those who could make change could be salesmen. We prepared for salesmanship through the use of real money. We became acquainted with the different denominations and had some practice in making change through problems suggested by the children and teacher. Addition and subtraction combinations

went up to par. The children worked in little groups, playing combination games, vieing with each other to see if they were strong enough to be chosen as salesmen.

"A listener caught many bits of interesting conversation on the day of the sale. One salesman became confused with the rush of trade. Another child while waiting her turn to become a saleswoman was watching affairs through the window. She was heard saying, 'Yes, Carl, give her two cents. She bought a flower pot for six cents and two packets of seeds, that was two cents. Give her two cents back 'cause she gave you a dime.'

"Another salesman had received a quarter. His sale had amounted to five cents. He was heard (at the change box) saying, 'Yes, I must give her twenty cents.' He began counting out twenty pennies, then quickly picked up two dimes.

"Another salesman was heard telling a small customer, 'This vase is three cents, that basket is five cents. The vase is nice and it is cheaper than the basket.'

"We had number work of a more formal type, following this experience, as sets of problems in addition and subtraction to be done by all the children within a limited time. The delight and enthusiasm shown at the close of one of these little contests was not surpassed on the ball field by their seniors, I am sure.

"We feel that our project touched every phase of our curriculum and that we all grew immeasurably through it. He who is busy and growing is always happy. Joy was one of our greatest outcomes."

Many of these activities will, of course, take the nature of projects; they will involve, for their successful completion, all the types of school work which first grade pupils do. The primary purpose, so far as arithmetic is concerned, is to provide a rich background of interesting, meaningful experiences upon which the number lessons can be based, and to afford a real motive for acquiring the fundamental number skills which are recommended for the first grade course of study. The formal presentation of the primary number facts can be made an ex-

tremely dull and uninteresting experience; on the other hand, it can be made a series of interesting and delightful experiences which will redound to the pupil's acquisition of subject matter and growth in skill.

THE COURSE FOR THE SECOND GRADE

Arithmetic will have a more prominent place in the second grade than it had in the first grade. It will be given a larger amount of time in the daily schedule. Assignments will be more specific and will cover a larger amount of subject matter. Both teacher and pupils will feel more definitely the responsibility of attaining certain minimum skills in the fundamentals.

The first responsibility of the second grade teacher is to see that each of her pupils has covered effectively the course provided for the first grade. This not only means that she must provide a thorough review of the first year's work but also that she must make careful provision for any of the first grade course which may have been omitted. The suggestions for a second-grade course which follow are based upon the assumption that all of the work suggested for the first grade has been carefully done.

The second grade course will be outlined under the same six topics which were used in the course for the first grade. They are: (1) counting and reading and writing numbers; (2) addition; (3) subtraction; (4) problems; (5) measurement; (6) activities.

1. **Counting and reading and writing numbers.**

(*a*) Enough counting by 1's, 5's, and 10's beyond 100 to demonstrate the continuation of the number system.
(*b*) Counting by 2's from 20 to 100.
(*c*) Reading and writing numbers to 1,000.
(*d*) Reading Roman numerals to XII.

Counting beyond 100 is not a frequent social experience; but it occurs on occasions. We plan that pupils shall count beyond

100 while reviewing the counting which they learned to do in the first grade and that they shall have just enough such experience to let them see how such counting is done—that they may see how they could count on through several hundreds, if they chose. Counting by 2's will be extended from 20 to 100 and applied to the counting of large groups of objects, such as seats (taking two rows at a time), shelves of books, etc.

If pupils have been taught to read and write numbers beyond 100 in the first grade, they will find the second grade assignment —reading and writing numbers to 1,000—quite easy. It is well to first count by hundreds, just as we counted by tens in learning to count to 100 (see page 24). The pupils will say and write, 100 200 300 400 500 600 700 800 900 1000, and then learn to fill in any designated gaps. Thus, the teacher may say, "Write all numbers from 647 to 662," etc.

In most courses of study, Roman notation to XII is taught in the first grade, and, consequently, one of the activities ordinarily listed for first grade pupils is telling time. We believe, however, that it is better to postpone telling time until the second grade is reached. We know of no scientific investigation of the subject but first grade teachers frequently report considerable difficulty in teaching their pupils to tell time. We have known many children of more than average intelligence who not only seemed to find it quite difficult to learn to tell time when in the first grade but who, furthermore, seemed to take no interest in the subject. To be sure, children vary in this respect, as in other respects, but there seem to be excellent arguments for postponing the study of the clock face until the second grade is reached and for teaching in the first grade some other topics which are commonly postponed to the second grade, as the first 25 of the 45 addition combinations.

If the schoolroom is not supplied with a clock having the Roman numerals, a large cardboard clock face, equipped with the two hands, should be furnished for practice in reading the

Roman numerals and in reading the clock face at various hypothetical times. Practice in telling time should not be limited to clock faces having Roman numerals, however, for many clocks and most watches have Arabic numerals.

2. **Addition.**

(*a*) The 20 combinations whose sums exceed 10.
(*b*) Single columns of three and four digits, without higher decade addition.
(*c*) Higher decade addition, sums not exceeding 39.
(*d*) Two and three addends of two digits each, no carrying.
(*e*) Two addends of three digits each, no carrying, sums not greater than 1000.
(*f*) Columns of four digits to practice higher decade addition to 39.
(*g*) The 19 zero facts in examples, but not as isolated facts.
(*h*) The terms, *add* and *sum*, and the signs, + and =.

Detailed suggestions for handling the addition work of this grade are given in Chapters IV and V. Our division of the work outlined in Chapter V between the second and the third grades is, of course, somewhat arbitrary but it is in harmony with the median practice of the better school systems.

The signs, + and =, are introduced in order that the equation form of statement may be used as a variety form in drill lessons. All combinations will first be presented in the vertical form and most of our practice on them will be given in that form since the vertical form is commonly used in written addition.

3. **Subtraction.**

(*a*) The remaining 20 combinations, those whose minuends are greater than 10.
(*b*) Examples having two and three digits in the minuend and in the subtrahend, no borrowing.
(*c*) Zeros in subtrahend and in remainder.
(*d*) The terms, *subtract*, *difference* and *remainder*, and the sign, —.

Methods for teaching subtraction have been given in Chapters IV and VI. It will be observed that we are providing that the second grade shall do any subtraction examples which do not involve numbers of more than three digits and which do not include borrowing.

4. **Problems.** Only the simpler and easier problems should be used. The operations employed should be limited to addition and subtraction and to the particular types of addition and subtraction which are taught in this grade. Sets of problems should be prepared for use with each of the kinds of examples taught. In addition, for instance, problems will give practice on the combinations, higher decade addition, single column addition, the addition of two addends of three digits each, zeros, etc.

Problem solving should occupy a rather prominent place in the second grade. In the third grade, the pupils will probably use a textbook containing hundreds of problems adapted to the skills developed in that grade. But in the second grade, pupils do not usually have a text, and the teacher, consequently, frequently follows the outline supplied by the course of study, trying to develop the fundamental skills which are prescribed, but may devote scant attention to the solution of problems. Problem solving deserves a much more prominent place in the second grade than it usually occupies.

5. **Measurement.**

(*a*) Time: year, hour, half-hour, quarter-hour, minute.
(*b*) Length: half-inch, quarter-inch, yard.
(*c*) Liquid: gallon, half-gallon.
(*d*) Money: half-dollar, quarter-dollar, dollar.
(*e*) Fractions: the meaning of one-fourth.
(*f*) Weight: pound, half-pound, quarter-pound.

It is a very common practice for courses of study to give, for the second grade, not only the measurement items recommended for that grade but also those which have been included in the course for the first grade. On the contrary, we have given, in the above outline, only the new measurement items which are to

be taught in grade two. It is intended, of course, that the second grade teacher shall provide for frequent review of, and practice on, the items of the preceding grade.

It has been provided that first grade children shall learn the meaning of the fraction, one-half. We are adding the fraction, one-fourth, for the second grade. It is not intended that the fraction form, $\frac{1}{4}$, shall be taught, but that the pupils shall understand the language and know, from much concrete experience, what one-fourth means. The relationship of one-fourth to quarter will then be indicated. Several very practical units of measurement can then be understood. These include the quarter-hour, the quarter-inch, the quarter-dollar, and the quarter-pound. The pupil will, of course, have practice in the most common language forms, saying, "quarter" as well as "quarter-dollar," "one-fourth of a pound" as well as a "quarter-pound," etc. Such expressions as "a quarter after four," "a quarter of six," etc., used in telling time, will then be meaningful.

Only the very common units of measurement, those which are likely to occur frequently in the experiences of second grade children and which they have not studied in the first grade, are included in the list of recommendations for this grade.

6. **Activities.** Almost all of the activities which have been mentioned in the course for grade one are also suitable for grade two. Certain modifications will be made to adapt the activities to the changing interest and increasing abilities of the children and to prevent loss of interest through monotony, but many of these activities are repeated throughout the years of a child's school life, occurring as routine parts of the day's work and may quite well be utilized in developing arithmetic knowledge and skill.

The following additional activities are suggestive of the kinds which may be used in the second grade.

(*a*) Weighing fellow members of the class.
(*b*) Measuring heights of pupils, using the appropriate apparatus.

(*c*) Keeping monthly records of heights and weights.
(*d*) Telling time.
(*e*) Looking up dates on the calendar, counting the number of weeks and months to the dates of interesting occurrences, etc.
(*f*) Looking up telephone numbers, reading and locating the license numbers of cars, finding house numbers.
(*g*) Reading the thermometer.
(*h*) Playing number games giving practice in the fundamentals of addition and subtraction.
(*i*) Finding the number of hours of sleep.
(*j*) Counting out and passing materials.
(*k*) Counting out classmates for games in and out of school.
(*l*) Comparing attendance records, counting the number who have been absent, the number who have been tardy, and the number who have neither been absent nor tardy.
(*m*) Keeping a record of the number of pages read in various books in a week.
(*n*) Finding differences between ages.
(*o*) Finding differences in the number of blocks pupils have to walk to school.
(*p*) Buying and making change; real and imaginary problems about costs, differences in prices, and amount received as change.

Most of the suggested activities need no comment; they are self-explanatory. A few, somewhat more difficult than others, are included because children are often interested in them and because they are very practical. Reading a thermometer, for example, is difficult because each mark usually means two degrees. This will give children excellent practice in counting by 2's from 40, 50, 60, etc.

Finding the number of hours of sleep involves subtracting the hour when the child goes to bed from 12 and adding the difference to the number of hours he sleeps after midnight. Thus: "Harry went to bed at 8 o'clock and got up at 7 o'clock. How many hours did he sleep?" Pupils may also find the number

of hours they are awake, the number of hours in the school day, etc. In this work, fractions should be avoided.

The list given will suggest many similar activities to the resourceful teacher, activities which may be much better adapted to a given local situation than are some of those in our list.

THE COURSE FOR THE THIRD GRADE

More than ten years ago, Jessup and Coffman[1] found that about one city in ten introduced a textbook in arithmetic before the third grade, that 56 per cent introduced a text in the third grade, and that the remainder of the 754 cities reporting—about one-third—first used a text in grades later than the third.

For many third grade teachers the course of study is largely the textbook. If a text has been adopted for the teacher's use, the course of study maker is frequently content to restate certain admonitions about drills, reviews, projects, problems, etc., already stated in the courses for the first and second grades, and to make detailed references to textbook pages for the work of the two semesters and for each month of the school year.

As one goes through courses of study in arithmetic, he is strongly impressed with the fact that in a large number of cases the particular topics taught, the amount of time allotted to each of these topics, and the order in which they occur are determined to a very large extent by the textbook which has been adopted for use. Indeed, in many so-called courses of study, the teacher who is responsible for the educational destinies of the fourth grade pupils receives no more specific instruction than that the assignment for the year begins on page 140 and continues to the end of the book.

Since textbooks show great variation in the topics included, the placing of those topics, and the extent and the method of the treatment of each, it is evident that courses of study which follow textbooks so closely must vary accordingly. Topics which

[1] Jessup, W. A., and Coffman, L. D. *The Supervision of Arithmetic.* The Macmillan Company, 1916, pp. 97-105.

are taught in the fourth grade in one school, may be taught in the fifth grade in another and in the sixth grade in a third. The teachers concerned, too, are frequently unaware of these great variations.

The more progressive school systems do not depend so much upon a textbook in determining what the course of study shall be. One finds here and there groups of expert teachers planning the work of each of the grades and then selecting two or more texts from which the lesson material may be obtained. The textbook thus becomes a source of information and practice material, as it should, and not a dictator of what the detailed course shall be.

In many communities, however, a single text will be the chief source of subject matter for the third grade. The determination of the topics to be taught and their sequence will usually rest in the hands of supervisors and other officials but, ordinarily, these will not differ radically from those in the prescribed text. The alert teacher, however, will use not one text but several and will select here and there the problems and examples which seem to suit her needs best. She will keep in step with the prescribed course but will not hesitate to eliminate portions of the adopted text in favor of better material on the same topics from other sources. Indeed, some courses of study make continuous reference to other texts than the one adopted for class use. The Detroit Course of Study in Arithmetic, 1924 edition, for example, refers to four different books in the third grade, one of which is the adopted text. Frequent references are made to specific pages in each of these four books.

The outline of suggested third grade material which follows is not designed to fit any particular text; indeed, it will probably fit none. Modifications will have to be made to suit local conditions and, perhaps, the adopted text but it is to be hoped that each school system and each teacher will be much more concerned with teaching children arithmetic than with following a prescribed textbook.

Following largely the plan used in outlining the courses for the two preceding grades, we shall suggest the content of the third grade course of study under nine topics: (1) reading and writing numbers; (2) addition; (3) subtraction; (4) multiplication; (5) division; (6) fractions; (7) problems; (8) measurement; (9) activities.

1. **Reading and writing numbers.**

   (*a*) Reading and writing numbers to 10,000.
   (*b*) Reading Roman numerals to XXX.

The extent to which instruction in Roman notation should be provided in the grades is a moot one. In Chapter I it was pointed out that the extent to which this subject was taught in the third grade varied from V in the Colorado state course to M in the course used in Norwalk, Connecticut. In the face of such variations in practice, one naturally hesitates to offer a specific recommendation for the course of any grade but it may be ventured that the statement of the following paragraph represents a conservative, but not too conservative point of view.

We can be sure that I to XII should be taught because of the frequency with which they occur on time pieces. These should be taught when the child learns to tell time; we have suggested that this be in the second grade. It is probably worth while to teach XIII to XXX for these are frequently used in textbooks in numbering chapters, in numbering figures and other illustrations, in numbering tables, and in numbering the pages which include the preface, table of contents and other matter at the beginning of a book. Beyond XXX uses for Roman numerals in the lives of most people are so infrequent that it does not seem to be worth while to make definite provision for them in the course of study. If a real need for them occurs, they can be learned when the need arises. It is important that the pupils also learn M for it is much used in handling certain commodities, as lumber. L, C, and D may also be taught as guideposts in the system, as it were. However, each of these latter should be taught when

the need arises and not in the third grade or in any other particular grade merely because it is thought that they may be needed at some indefinite time in the future. Rarely do pupils have to write Roman numerals, except the very small ones. They need merely to be able to read them and understand them when they are encountered.

2. **Addition.**

(*a*) Higher decade combinations, sums not exceeding 39.
(*b*) Higher decade combinations needed in carrying in multiplication.
(*c*) Carrying.
(*d*) Single columns not to exceed seven addends (except in teaching multiplication combinations).
(*e*) Two-column examples, limited to six addends.
(*f*) Three-column examples, limited to five addends.
(*g*) Addends of unequal length, no addend to have more than three digits.
(*h*) Checking.

Each of these topics has been fully discussed in the chapter on the teaching of addition (Chapter V).

3. **Subtraction.**

(*a*) Borrowing.
(*b*) Zeros in minuend.
(*c*) Examples limited to four-digit minuends and four-digit subtrahends.
(*d*) Checking.

For a discussions of these topics, see Chapter VI.

4. **Multiplication.**

(*a*) The 45 combinations, to 9×9.
(*b*) Examples without carrying, limited to one-digit multipliers and two-digit multiplicands.
(*c*) Examples with carrying, limited to one-digit multipliers and three-digit multiplicands.
(*d*) Zeros in the multiplicand.
(*e*) Checking.
(*f*) The sign, $\times$, and the terms, *multiply* and *product*.

For a discussion of these topics, see Chapters VII and VIII.

5. **Division.**

(*a*) The 45 combinations, corresponding to the 45 multiplication combinations.
(*b*) Examples having one-digit divisors and two-digit quotients, no carrying.
(*c*) The primary facts with remainders.
(*d*) Examples involving carrying,, limited to one-digit divisors and three-digit quotients, with and without remainders.
(*e*) Zeros in two- and three-digit quotients, with and without remainders, work limited to one-digit divisors.
(*f*) All work will be done by long division form.
(*g*) Checking.
(*h*) The sign, $\div$, and the terms, *divide* and *quotient.*

Detailed suggestions on the teaching of division are provided in Chapter IX.

6. **Fractions.** Fractions are conveniently introduced in connection with elementary work in division. When the pupil has learned that four 5's are 20, he easily learns and understands that $\frac{1}{4}$ of 20 is 5 and that $\frac{1}{5}$ of 20 is 4. Since we teach the multiplication and division facts to 9×9 and to $81 \div 9$, we can easily introduce the fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, and $\frac{1}{9}$. Each of these should be used by the third grade pupils as a means of providing a varied form of statement in drill lessons and as a means of making the multiplication and division facts more meaningful, but, in actual life experiences, some of these fractions are of very rare occurrence while others occur quite frequently. It seems best, then, to give scant attention to sixths, sevenths, and ninths and to give less attention to fifths and eighths than to halves, thirds, and fourths.

Third grade pupils should have much practice on $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, applied to concrete objects and to numbers. The fractions, $\frac{2}{3}$ and $\frac{3}{4}$, may also be introduced through objective representation and used in connection with smaller numbers, as $\frac{2}{3}$ of 6, $\frac{3}{4}$ of 8, etc. The forms, $\frac{2}{3}$ and $\frac{3}{4}$, may quite well be postponed

to the fourth grade, however, in the event that there is insufficient time for them.

7. **Problems.** Problems involving the types of examples which have been taught in all four operations will be used in abundance in the third grade. No textbook contains a sufficient number of problems suitable for third grade pupils. Supplementary assignments in problem solving may be drawn from the experiences of the pupils in and out of school and from other texts.

Third grade pupils will enjoy solving problems involving two operations. These are of practical value for they frequently occur in their out-of-school experiences. For example, "Mary earned 75 cents taking care of a neighbor's baby. If she takes two of her friends to the picture show and buys three tickets at 15 cents each, how much will she have left?" is a problem of the type with which third-graders are frequently concerned.

8. **Measurement.** In addition to a review of, and further practice in the units of measure taught in the first and second grades, the following may be added in grade three.

(*a*) Length: mile.
(*b*) Weight: ounce.
(*c*) Abbreviations of measures taught.
(*d*) Use of fractions in comparing units of measure, as a pint is one-half of a quart, etc.

Denominate measures afford an excellent opportunity for further practical work with the more common fractions. Pupils like to compare pints with quarts, quarts with gallons, etc. This work also ties up well with multiplication and division. Pupils will change weeks to days and days to weeks, quarts to gallons and gallons to quarts, inches to feet and feet to inches, etc. Excellent problems involving multiplication and division can be prepared in this manner.

9. **Activities.** Many of the activities suggested as suitable for the second grade course will also be useful in the third grade; indeed, some of those listed in the course for the first grade will be continued through the second, third and later grades. In addi-

tion to the suggestions offered for the first and second grades, the following are given as particularly well adapted to grade three.

(*a*) Earning and spending money. Planning Christmas expenditures, etc.
(*b*) Ordering from a catalog.
(*c*) Planning parties.
(*d*) Planning a school picnic.
(*e*) Making birthday books containing dates of birthdays of pupils in the room.
(*f*) Finding the total number of hours in a week that each pupil who studies music practises; comparing records.
(*g*) Computing and comparing the costs of classified advertisements, at 3¢ a word, etc.
(*h*) Planning a trip: expenditures for gasoline, meals, hotel, etc.
(*i*) Making estimates: estimating distances, weights, heights, the number of people at a school entertainment, the number of cars which pass the school in an hour, etc.
(*j*) Computing the cost of a new outfit of clothing for a boy or girl.
(*k*) Taking an inventory of the number of supplementary readers, other reference books, chairs, erasers, etc., in the room and estimating the number in the building, assuming that each room has the same number (number of rooms not to exceed nine).
(*l*) Finding the total school enrollment by adding the enrollments of the various rooms.
(*m*) Comparing heights and weights: finding the difference in inches between tallest and shortest, the difference in pounds between the heaviest and the lightest, etc.
(*n*) Playing number games, giving practice on the fundamentals of multiplication and division.
(*o*) "Making the school garden pay."

The last activity is taken from the Twentieth Yearbook of the National Society for the Study of Education (page 35), one of the references given at the end of the chapter. This activity was reported by Ellice E. Burk of Cleveland, as follows:

"The seeds for the garden were provided by the school and the garden was a class project. The children did all the work with the exception of plowing. When the vegetables were ready for the market, the class was divided into committees, each one having a particular responsibility, such as keeping in touch with the market prices, soliciting customers or delivering goods. At regular intervals the crops were harvested, weighed, measured, put in bags with labels attached giving quantity, prices, and name of customers. The arithmetic lesson for that day consisted in finding out how much the sales amounted to and of keeping the record in good form. A strict account was kept, and at the end of the season, after paying back the amount advanced for seeds, the class had a nice little sum which was spent in buying some much desired apparatus."

STANDARDS OF ATTAINMENT

Throughout the preceding chapters, emphasis has been placed on the point of view that the habits which pupils acquire in the fundamental operations of arithmetic must be economical habits, that their responses to the primary facts of the four operations must be immediate and direct, and that all indirect and roundabout methods of arriving at sums, differences, products, and quotients must be carefully avoided. If these precepts have been followed from the time of the pupil's first instruction in number, there will be little need for setting up standards of attainment in speed and accuracy; they will take care of themselves. Speed in itself is an important dimension of a pupil's ability in arithmetic only in so far as it is a symptom of mastery and the effective operation of fundamental bonds. If we have reason to believe that the pupil's habits in the fundamentals of arithmetic are correct, the less attention given to speed in the primary grades, the better.

If a pupil's habits are right, the time which he takes to respond to the stimulus, "How many are five 8's?" will depend upon his own reaction-time, assuming that other conditions, such as free-

dom from distraction, are equal. To force the deliberate individual to greater speed will probably disturb the operation of his bonds, leading to a loss of self-control and to errors. It seems wise to measure the progress of pupils through the use of tests, in which both speed and accuracy are determined, and it is well to continue practice until a reasonable goal is reached, but it seems unwise to urge upon children the importance of more speed. The only satisfactory standard in accuracy is one hundred per cent. The only satisfactory speed standard is that which can be attained with a reasonable amount of practice *without loss in accuracy.* What constitutes a reasonable amount of practice is not very well known.

Perhaps as defensible a standard as any for pupils to attain by the end of the third year is that set by the Denver course of study in arithmetic. It is expressed as follows:

1. Addition
   - (*a*) Fifteen single columns of six figures each in 5 minutes.
   - (*b*) Ten exercises of five two-figured numbers in 5 minutes.
   - (*c*) Seven exercises of four three-figured numbers in 5 minutes.
2. Subtraction
   Twelve exercises of three-figured numbers, not more than one "carrying" in 5 minutes.
3. Multiplication
   Twelve exercises of three-figured numbers by a one-figured number in 5 minutes.
4. Division
   Twelve exercises of a four-figured number divided by a one-figured number in 5 minutes.

With the development of the measurement movement and the improvement of methods of instruction, however, standards of attainment must continually vary. What seems best this year may seem too high or too low, or otherwise poor, next year. If the teacher will see that the primary bonds are formed well, will skillfully distribute interesting forms of practice over the fun-

damental facts, will occasionally measure her pupils' progress, and will permit her pupils to find many interesting and useful applications for their skills in problems and other activities, she need not fear but that her pupils, if normal, will learn arithmetic and learn it well.

EXERCISES

1. What importance do you attach to the course of study in the primary grades?

2. Why should teachers participate in the making of a course of study?

3. What are the characteristics of a good course of study?

4. Why is continuous revision of the course of study necessary?

5. How would you differentiate the course of study to suit the varying abilities of a group of pupils?

6. Summarize the arguments for including arithmetic in the first grade.

7. What topics are recommended for inclusion in the first grade course?

8. In many courses the addition combinations are postponed until the second grade is reached. Summarize the argument for and against including addition combinations in the course for the first grade.

9. To what extent should the second or third grade teacher provide for a review of the first year's work?

10. Why do we not recommend the teaching of Roman notation in the first grade?

11. How does the work in measurement facilitate the teaching of the meaning of certain fractions?

12. To what extent should the adopted textbook constitute the course of study?

13. How much Roman notation should be taught in the third grade?

14. Discuss the place of standards of attainment in the primary grades.

15. To what extent is speed important in the fundamental operations?

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